Estimating robustness – Online appendix

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1 Discrete sampling from the continuous-time model

The state vector X_t follows a multivariate Ornstein-Uhlenbeck process, so it has a conditional normal distribution for all t. Define the unconditional mean as $\mu \equiv \kappa^{-1}\phi$. The SDE is

$$dX_t = -\kappa\left(X_t - \mu\right)dt + \sigma dW_t$$

For an arbitrary $\tau \geqslant 0$, the distribution of $X_{t+\tau}$, conditional on X_t, follows

$$X_{t+\tau}|X_t \sim \mathcal{N}\left(\bar{x}_{t+\tau},\ \ \Sigma_\tau\right)$$

with

$$\bar{x}_{t+\tau} = (I - \exp(-\kappa\tau))\mu + \exp(-\kappa\tau)X_t \qquad \text{vec}(\Sigma_\tau) = (\kappa\oplus\kappa)^{-1}\left(I - \exp(-(\kappa\oplus\kappa)\tau)\right)\text{vec}(\sigma\sigma^T)$$

At the same time, the additive functional vector $d\ln Y_t$ follows

$$d\log Y_t = \left(\beta_0 + \beta_1 X_t\right)dt + \alpha dW_t$$

hence

$$\log Y_{t+\tau} - \log Y_t = \beta_0\tau + \beta_1\int_t^{t+\tau} X_s ds + \int_t^{t+\tau}\alpha dW_t = \left(\beta_0 + \beta_1\mu\right)\tau + \int_t^{t+\tau}\beta_1\exp\left(-\kappa s\right)\left(X_t - \mu\right)ds +$$
$$+ \int_t^{t+\tau}\beta_1\left[\int_t^s \exp\left(-\kappa(s-u)\right)\sigma dW_u\right]ds + \int_t^{t+\tau}\alpha dW_s$$

This implies that the conditional distribution of $\log Y_{t+\tau} - \log Y_t$ is normal with **mean**

$$\mathbb{E}\left[\log Y_{t+\tau} - \log Y_t \mid X_t = x\right] = \left(\beta_0 + \beta_1\mu\right)\tau + \beta_1\kappa^{-1}\left(I - \exp(-\kappa\tau)\right)[x - \mu] =$$
$$= \underbrace{\beta_0\tau + \beta_1\left(I - \kappa^{-1} + \kappa^{-1}\exp(-\kappa\tau)\right)\mu\tau}_{\equiv\beta_{0D}} + \underbrace{\beta_1\kappa^{-1}\left(I - \exp(-\kappa\tau)\right)}_{\equiv\beta_{1D}}x$$

The **covariance matrix** of $\log Y_{t+\tau} - \log Y_t$ given X_t comes from the exposure to $\int_t^{t+\tau} dW_s$

$$\beta_1 \int_t^{t+\tau} \left[\int_t^s \exp(-\kappa(s-u))\sigma dW_u \right] ds + \int_t^{t+\tau} \alpha dW_s$$

The first term can be rewritten as

$$\begin{aligned}\beta_1 \int_t^{t+\tau} \left[\int_t^s \exp(-\kappa(s-u))\sigma dW_u \right] ds &= \beta_1 \int_t^{t+\tau} \left[\int_u^{t+\tau} \exp(-\kappa(s-u))\sigma ds \right] dW_u = \\ &= \beta_1 \int_t^{t+\tau} \left(I - \exp(\kappa(u-[t+\tau]))\right) \kappa^{-1}\sigma dW_u\end{aligned}$$

Therefore the covariance matrix is determined by the term

$$\int_t^{t+\tau} \left[\beta_1 \kappa^{-1}\sigma + \alpha - \beta_1 \exp(\kappa(u-[t+\tau]))\kappa^{-1}\sigma\right] dW_u$$

Using this with the Ito isometry we arrive at the covariance matrix

$$\begin{aligned}\left(\beta_1\kappa^{-1}\sigma + \alpha\right)\left(\alpha^T + \sigma^T(\kappa^T)^{-1}\beta_1^T\right)\tau - \int_t^{t+\tau} \left[\alpha + \beta_1\kappa^{-1}\sigma\right]\sigma^T(\kappa^T)^{-1}\exp(\kappa^T(u-[t+\tau]))\beta_1^T du \\ - \int_t^{t+\tau} \beta_1 \exp(\kappa(u-[t+\tau]))\kappa^{-1}\sigma\left[\alpha^T + \sigma^T(\kappa^T)^{-1}\beta_1^T\right] du \\ + \int_t^{t+\tau} \beta_1 \exp(\kappa(u-[t+\tau]))\kappa^{-1}\sigma\sigma^T(\kappa^T)^{-1}\exp(\kappa^T(u-[t+\tau]))\beta_1^T du\end{aligned}$$

the second and third term can be written as, respectively

$$\begin{aligned}\left[\alpha + \beta_1\kappa^{-1}\sigma\right]\sigma^T(\kappa^T)^{-1}\left(I - \exp(-\kappa^T\tau)\right)(\kappa^T)^{-1}\beta_1^T \quad \text{and} \\ \beta_1\kappa^{-1}\left(I - \exp(-\kappa\tau)\right)\kappa^{-1}\sigma\left[\alpha^T + \sigma^T(\kappa^T)^{-1}\beta_1^T\right]\end{aligned}$$

while the fourth term is of the form (without β_1 in the front and the end)

$$T4(\tau) = \left[\int_0^\tau \exp(-\kappa s) Q_c \exp(-\kappa^T s) ds\right]$$

where $Q_c = \kappa^{-1}\sigma\sigma^T(\kappa^T)^{-1}$ is positive semi-definite symmetric matrix. We can evaluate this integral by using the formula

$$\text{vec}(T4(\tau)) = (\kappa \oplus \kappa)^{-1}\left(I - \exp(-(\kappa \oplus \kappa)\tau)\right)\text{vec}(Q_c)$$

2 Yield curve

The price of a zero-coupon bond with maturity τ is

$$P_t^{(\tau)}(x) = \widetilde{\mathbb{E}}\left[\exp(-\delta\tau)\frac{(CP)_t}{(CP)_{t+\tau}} \mid X_t = x\right] = \widetilde{\mathbb{E}}\left[\exp(-\delta\tau)\frac{(CP)_0}{(CP)_\tau} \mid X_0 = x\right]$$

The second equality follows from the fact that since both C and P are multiplicative functionals, their product is also a multiplicative functional. This implies that the equilibrium yield curve is time-invariant and it hinges only on the current value of X. Using the given law of motions for C and P, we can write

$$
\begin{aligned}
P^{(\tau)}(x) &= \widetilde{\mathbb{E}}\left[\exp\left(-\delta\tau - \int_0^\tau d\log C_s - \int_0^\tau d\log P_s\right) \mid X_0 = x\right] = \\
&= \widetilde{\mathbb{E}}\left[\exp\left(\int_0^\tau \left[-\delta - \iota_2 \cdot \beta_0 - \iota_2^T \beta_1 X_s\right] ds + \int_0^\tau -\iota_2^T \alpha dW_s\right) \mid X_0 = x\right]
\end{aligned}
$$

Let the exponential inside the expectation operator be M_t, and translate everything into the notation of Borovička et al. (2011), to derive ODEs for $a(\tau)$ and $b(\tau)$ in the expression of the the expectation of a multiplicative functional with affine dynamics, i.e. $\widetilde{\mathbb{E}}[M_t \mid X_0 = x] = \exp(a(t) + b(t)x)$. The dictionary of corresponding notations is

$$
\beta(x) = \bar{\beta}_0 + \bar{\beta}_1 \cdot (x - \bar{x}) = \left[-\delta - \iota_2 \cdot \widetilde{\beta}_0 - \iota_2^T \widetilde{\beta}_1 \bar{x}\right] + \left[-\widetilde{\beta}_1^T \iota_2\right] \cdot (x - \bar{x}) \qquad \bar{\alpha} = -\alpha^T \iota_2
$$

where the Markov state vector has the following drift and volatility terms

$$
\mu(x) = \bar{\mu}_{11}(x - \bar{x}) = \widetilde{\phi} - \widetilde{\kappa} x = (-\widetilde{\kappa})\left[x - \widetilde{\kappa}^{-1}\widetilde{\phi}\right] \qquad \bar{\sigma}_1 = \sigma
$$

where $\widetilde{\kappa} = \kappa - \sigma\eta_1$. Then the ODEs are

$$
\begin{aligned}
\frac{d}{dt} b(t)^T &= \bar{\beta}_1^T + b(t)^T \bar{\mu}_{11} \\
\frac{d}{dt} a(t) &= \bar{\beta}_0 - \left[\bar{\beta}_1^T + b(t)^T \bar{\mu}_{11}\right] \bar{x} + \frac{1}{2}\left(\bar{\alpha}^T \bar{\alpha} + 2\bar{\alpha}^T \bar{\sigma}_1^T b(t) + b(t)^T \bar{\sigma}_1 \bar{\sigma}_1^T b(t)\right)
\end{aligned}
$$

Using the notation of the main text, this becomes

$$
\begin{aligned}
\frac{d}{dt} b(t)^T &= -\iota_2^T \beta_1 + b(t)^T(-\widetilde{\kappa}) \\
\frac{d}{dt} a(t) &= -\delta - \iota_2 \cdot \widetilde{\beta}_0 - b(t)^T(-\widetilde{\kappa})\bar{x} + \frac{1}{2}\left(\iota_2^T \alpha\alpha^T \iota_2 - 2\iota_2^T \alpha\sigma^T b(t) + b(t)^T \sigma\sigma^T b(t)\right)
\end{aligned}
$$

3 Calculating Chernoff entropy

Relative to the benchmark model we derive alternative models through martingales $\frac{dZ^H}{Z_t^H} = H_t \cdot dW_t$ and parametrize them by their exposure process H. Since H_t takes the form $H_t = \eta(X_t)$, the alternative models are Markovians. Given the process H, Chernoff entropy is defined as

$$
\chi(Z_t^H, x) = -\inf_{0 \leqslant \mathbf{r} \leqslant 1} \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}\left[(Z_t^H)^{\mathbf{r}} | X_0 = x\right]
$$

Both Z_t^H and $\left(Z_t^H\right)^{\mathbf{r}}$ are multiplicative functionals, so apart from the inf over $\mathbf{r}$, χ is equal to the long-term growth rate (provided it is well-defined) of the implied multiplicative semigroup. Hansen and Scheinkman (2009) tell us how to calculate this growth rate using the principal eigenvalue problem

$$\mathbb{E}\left[\left(Z_t^H\right)^{\mathbf{r}} e(X_t)|X_0 = x\right] = \exp(-\psi t)e(x)$$

for all t where e is strictly positive. The local counterpart is

$$[\mathbb{G}e](x) = \lim_{\tau\downarrow 0} \frac{\mathbb{E}\left[\left(Z_\tau^H\right)^{\mathbf{r}} e(X_\tau)|X_0 = x\right] - e(x)}{\tau} = -\psi e(x)$$

so we need the drift of $\left(Z_t^H\right)^{\mathbf{r}} e(X_t)$. Ito's lemma implies

$$d\left[\left(Z_t^H\right)^{\mathbf{r}} e(X_t)\right] = \frac{\mathbf{r}(\mathbf{r}-1)}{2}\left(Z_t^H\right)^{\mathbf{r}} |H_t|^2 e(X_t) + \left[\nabla e(X_t)^T(\phi - \kappa X_t) + \frac{1}{2}\text{tr}\left(\sigma^T e_{xx}(X_t)\sigma\right)\right]\left(Z_t^h\right)^{\mathbf{r}} + \\ + \mathbf{r}\left(Z_t^h\right)^{\mathbf{r}} \nabla e(X_t)^T \sigma H_t$$

evaluating this at $t = 0$ yields

$$[\mathbb{G}e](x) = \frac{\mathbf{r}(\mathbf{r}-1)}{2}|H_t|^2 e(x) + \left[\nabla e(X_t)^T(\phi - \kappa X_t) + \frac{1}{2}\text{tr}\left(\sigma^T e_{xx}(X_t)\sigma\right)\right] + \mathbf{r}\nabla e(X_t)^T\sigma H_t$$

$$-\psi = \frac{\mathbf{r}(\mathbf{r}-1)}{2}|H_t|^2 + \nabla \log e(x)^T(\underbrace{\phi + \mathbf{r}\sigma\eta_0}_{\equiv\tilde{\phi}} - \underbrace{\kappa - \mathbf{r}\sigma\eta_1]}_{\equiv\tilde{\kappa}} x) + \frac{1}{2}\text{tr}\left(\sigma^T \frac{1}{e(x)} e_{xx}(x)\sigma\right)$$

If $h(X_t)$ is affine in X_t, then guess and verify shows that the eigenfunction is given by the exponential of a quadratic function of X_t. Using the form $\log e(x) = \lambda_0 + \lambda_1 \cdot x + x^T\lambda_2 x$, we get

$$\begin{aligned}\nabla \log e(x)^T &= \lambda_1^T + x^T\left(\lambda_2^T + \lambda_2\right) \\ \frac{1}{e(x)} e_{xx}(x) &= \left(\lambda_2^T + \lambda_2\right) + \left[\lambda_1 + \left(\lambda_2 + \lambda_2^T\right)x\right]\left[\lambda_1^T + x^T\left(\lambda_2^T + \lambda_2\right)\right]\end{aligned}$$

and

$$\frac{1}{2}\text{tr}\left(\sigma^T\frac{1}{e(x)}e_{xx}(x)\sigma\right) = \frac{1}{2}\text{tr}\left(\sigma^T\left(\lambda_2^T + \lambda_2 + \lambda_1\lambda_1^T\right)\sigma\right) + x^T\left(\lambda_2^T + \lambda_2\right)\sigma\sigma^T\lambda_1 + \\ + \frac{1}{2}x^T(\lambda_2^T + \lambda_2)\sigma\sigma^T(\lambda_2^T + \lambda_2)x$$

and the ODE becomes

$$-\psi = \frac{\mathbf{r}(\mathbf{r}-1)}{2}|H_t|^2 + \left(\lambda_1^T + x^T(\lambda_2^T + \lambda_2)\right)(\phi - \kappa x + \mathbf{r}\sigma H_t) + \frac{1}{2}x^T(\lambda_2^T + \lambda_2)\sigma\sigma^T(\lambda_2^T + \lambda_2)x + \\ + \frac{1}{2}\text{tr}\left(\sigma^T\left(\lambda_2^T + \lambda_2 + \lambda_1\lambda_1^T\right)\sigma\right) + x^T\left(\lambda_2^T + \lambda_2\right)\sigma\sigma^T\lambda_1$$

Matching coefficients for x^2 yields

$$0 = \frac{\mathbf{r}(\mathbf{r}-1)}{2}\eta_1^T\eta_1 - \tilde{\kappa}^T\lambda_2 - \lambda_2\tilde{\kappa} + 2\lambda_2\sigma\sigma^T\lambda_2$$

which is a continuous time algebraic Riccati equation of the form

$$A^T\lambda_2 + \lambda_2 A - \lambda_2 B R^{-1} B^T \lambda_2 + Q = 0$$

where

$$A = \widetilde{\kappa} = \kappa - \mathbf{r}\sigma\eta_1 \qquad B = \sqrt{2}\sigma \qquad R = -I_K$$

Matching coefficients for x^T

$$\begin{aligned} 0 &= \mathbf{r}(\mathbf{r}-1)\eta_1^T\eta_0 + (\lambda_2^T + \lambda_2)\widetilde{\phi} - \left(\widetilde{\kappa} - (\lambda_2^T + \lambda_2)\sigma\sigma^T\right)\lambda_1 \\ \lambda_1 &= \left(\widetilde{\kappa} - (\lambda_2^T + \lambda_2)\sigma\sigma^T\right)^{-1}\left[\mathbf{r}(\mathbf{r}-1)\eta_1^T\eta_0 + (\lambda_2^T + \lambda_2)\widetilde{\phi}\right] \end{aligned}$$

The scalar terms can be solved for ψ

$$-\psi(\mathbf{r}) = \frac{\mathbf{r}(\mathbf{r}-1)}{2}\eta_0\cdot\eta_0 + \lambda_1^T\widetilde{\phi} + \frac{1}{2}\mathrm{tr}\left(\left(\lambda_2^T + \lambda_2 + \lambda_1\lambda_1^T\right)\sigma\sigma^T\right)$$

Maximizing $\psi(\mathbf{r})$ over $[0,1]$ w.r.t $\mathbf{r}$ gives the Chernoff entropy associated with H. The half-life then can be calculated as

$$HL = \frac{\log(2)}{\max\limits_{\mathbf{r}\in[0,1]}\psi(\mathbf{r})}$$

4 Relationship between robust control and recursive utility

Recursive utility model with IRS= 1: See Section 5 in Borovička et al. (2011). The stochastic discount factor is multiplicative, the last term is a martingale, but the exposure of the martingale is not state dependent. More precisely, the recursive utility SDF is

$$S_t = \exp\left(-\delta t - \iota\cdot\beta_0 t - \iota'\beta_1\int_0^t X_s ds - \frac{|\widetilde{\alpha}|^2}{2}t - [\iota'\alpha - \widetilde{\alpha}']W_t\right)$$

with $\widetilde{\alpha} = (1-\gamma)\left[\frac{1}{\delta}\alpha' e_1 + \sigma'\bar{v}_1\right] = (1-\gamma)\left[\frac{1}{\delta}\alpha' e_1 + \sigma'\nabla_x W\right]$. Where W is the linear value function with

$$\nabla W_x = \bar{v}_1 = \left[\delta I_2 - (-\kappa)'\right]^{-1}\frac{\beta_1' e_1}{\delta}$$

Using Ito's lemma we arrive at

$$\frac{dS_t}{S_t} = -\left(\delta + \iota\cdot\beta_0 + \iota'\beta_1 X_t - \frac{|\iota'\alpha|^2}{2} + (\alpha'\iota)\cdot\widetilde{\alpha}\right)dt - [\alpha'\iota - \widetilde{\alpha}]\cdot dW_t$$

Robust preference model: The stochastic discount factor is multiplicative, the last term is a martingale with exposure $\eta(X)$. The SDF is

$$S_t = \exp\left(-\delta t - \log(CP)_t - \log(CP)_0 - \frac{|\eta(X_t)|^2}{2}t - [\iota\alpha - \eta(X_t)]W_t\right)$$

with

$$\eta(X) = -\frac{1}{\ell}\left[\frac{1}{\delta}\alpha' e_1 + \sigma' 2(\bar{v}_{1,1}\ell + \bar{v}_{1,0}) + \sigma' 2\bar{v}_2 \ell X_t\right] \equiv \eta_0 + \eta_1 X_t$$

Using Ito's lemma

$$\frac{dS_t^{nom}}{S_t^{nom}} = -\left(\delta + \iota\cdot\beta_0 + \iota'\beta_1 X_t - \frac{|\iota'\alpha|^2}{2} + (\alpha'\iota)\cdot\eta(X_t)\right)dt - \left(\alpha'\iota - H_t^*\right)\cdot dW_t$$

Special case: $\xi_1 = 0$ and $\xi_2 = \mathbf{0}$. In this case $\bar{v}_2 = 0$ and $\bar{v}_{1,1} = 0$, hence

$$\bar{v}_{1,0} = \frac{1}{2\delta}\left[\delta I_2 - (-\kappa)'\right]^{-1}\beta_1' e_1$$

$$\ell^* = \sqrt{\frac{\bar{v}_{0,-1}}{\bar{v}_{0,1}}} = \sqrt{\frac{\left|\frac{1}{\delta}\alpha' e_1 + 2\sigma'\bar{v}_{1,0}\right|^2}{\xi_0}} = \sqrt{\frac{1}{\xi_0}}\left|\frac{1}{\delta}\alpha' e_1 + 2\sigma'\bar{v}_{1,0}\right|$$

Observational equivalence is obtained when

$$(\gamma - 1) = \frac{1}{\ell^*(\xi_0)} = \sqrt{\frac{\xi_0}{\left|\frac{1}{\delta}\alpha' e_1 + 2\sigma'\bar{v}_{1,0}\right|^2}}$$

5 Calculating standard errors for the parameters of ξ

First stage: The parameter is a 11-vector $\theta \equiv (\kappa, \sigma, \alpha)$

$$\max_{\theta}\ \frac{1}{T}\sum_{t=1}^{T}\log L(Y_t|Y^{t-1};\theta)$$

then

$$g_1(\theta) = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial \log L(Y_t|Y^{t-1};\theta)}{\partial\theta}$$

and

$$-G_{11} = -\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 \log L(Y_t|Y^{t-1};\theta)}{\partial\theta^2} \to \mathcal{I}_{FI}$$

Second stage: the objective function is

$$\min_{\xi} \quad \frac{1}{T}\frac{1}{N}\sum_{t=1}^{T}\sum_{\tau=1}^{N}(\widehat{y}_t^{(\tau)}(\theta,\xi) - y_t^{(\tau)})^2$$

where

$$\widehat{y}_t^{(\tau)}(\theta,\xi) = a_\tau(\theta,\xi) + b_\tau(\theta,\xi)\cdot\widehat{X}_t(\theta)$$

The corresponding Jacobian and Hessian can be written as

$$\begin{aligned}
\frac{\partial\widehat{y}_t^{(\tau)}(\theta,\xi)}{\partial\xi} &= \frac{\partial a_\tau(\theta,\xi)}{\partial\xi} + \frac{\partial b_\tau(\theta,\xi)\cdot X_t(\theta)}{\partial\xi} \\
\frac{\partial\widehat{y}_t^{(\tau)}(\theta,\xi)}{\partial\theta} &= \frac{\partial a_\tau(\theta,\xi)}{\partial\theta} + \left(\frac{\partial b_\tau(\theta,\xi)}{\partial\theta}\right)' X_t(\theta) + b_\tau(\theta,\xi)'\frac{\partial X_t(\theta)}{\partial\theta} \\
\frac{\partial^2\widehat{y}_t^{(\tau)}(\theta,\xi)}{\partial\xi^2} &= \frac{\partial^2 a_\tau(\theta,\xi)}{\partial\xi^2} + \frac{\partial^2 b_\tau(\theta,\xi)\cdot X_t(\theta)}{\partial\xi^2} \\
\frac{\partial^2\widehat{y}_t^{(\tau)}(\theta,\xi)}{\partial\xi\partial\theta} &= \frac{\partial^2 a_\tau(\theta,\xi)}{\partial\xi\partial\theta} + \sum_i \frac{\partial^2 b_\tau^i(\theta,\xi)}{\partial\xi\partial\theta}X_t^i(\theta) + \left(\frac{\partial b_\tau(\theta,\xi)}{\partial\xi}\right)'\frac{\partial X_t(\theta)}{\partial\theta}
\end{aligned}$$

The FOC of the objective function is

$$g_2(\widehat{\theta},\widehat{\xi}) = 2\frac{1}{T}\frac{1}{N}\sum_{t=1}^{T}\sum_{\tau=1}^{N}\left(\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi}) - y_t^{(\tau)}\right)\frac{\partial\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi})}{\partial\xi} = 0$$

and the Jacobian of g_2 is

$$\begin{aligned}
G_{2,1} &= 2\frac{1}{T}\frac{1}{N}\sum_{t=1}^{T}\sum_{\tau=1}^{N}\left[\frac{\partial\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi})}{\partial\xi}\left(\frac{\partial\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi})}{\partial\theta}\right)' + \left(\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi}) - y_t^{(\tau)}\right)\frac{\partial^2\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi})}{\partial\xi\partial\theta}\right] \\
G_{2,2} &= 2\frac{1}{T}\frac{1}{N}\sum_{t=1}^{T}\sum_{\tau=1}^{N}\left[\frac{\partial\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi})}{\partial\xi}\left(\frac{\partial\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi})}{\partial\xi}\right)' + \left(\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi}) - y_t^{(\tau)}\right)\frac{\partial^2\widehat{y}_t^{(\tau)}(\widehat{\theta},\widehat{\xi})}{\partial\xi\partial\xi'}\right]
\end{aligned}$$

Expand $g_1(\theta)$ and $g_2(\theta,\xi)$ around the true parameter values θ_0 and ξ_0 and multiply by $\sqrt{T}$

$$\begin{aligned}
\sqrt{T}g_1(\widehat{\theta}) &= \sqrt{T}g_1(\theta_0) + G_{11}\sqrt{T}(\widehat{\theta}-\theta_0) \\
\sqrt{T}g_2(\widehat{\theta},\widehat{\xi}) &= \sqrt{T}g_2(\theta_0,\xi_0) + G_{21}\sqrt{T}(\widehat{\theta}-\theta_0) + G_{22}\sqrt{T}(\widehat{\xi}-\xi_0)
\end{aligned}$$

Given that the LHS variables are equal to zero by construction, we can rewrite this system as

$$\begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix}\sqrt{T}(\widehat{\theta}-\theta_0) \\ \sqrt{T}(\widehat{\xi}-\xi_0)\end{bmatrix} = -\sqrt{T}\begin{bmatrix} g_1(\theta_0) \\ g_2(\theta_0,\xi_0)\end{bmatrix}$$

and using the inversion formula for blocked matrices

$$\begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (G_{11})^{-1} & 0 \\ -(G_{22})^{-1}G_{21}(G_{11})^{-1} & (G_{22})^{-1} \end{bmatrix}$$

we can write the asymptotic variances as

$$\widehat{\mathbb{V}}\left(\sqrt{T}(\widehat{\theta} - \theta_0)\right) = (G_{11}^{-1})\mathcal{I}_{FI}(G_{11}^{-1})' \to \mathcal{I}_{FI}^{-1} \approx \left(-\frac{1}{T}\sum \frac{\partial^2 \log L}{\partial \theta^2}\right)^{-1}$$

$$\widehat{\mathbb{V}}\left(\sqrt{T}(\widehat{\xi} - \xi_0)\right) = \left[-(G_{22a})^{-1}G_{21}G_{11}^{-1} \quad G_{22}^{-1}\right]\Omega_T\left[-G_{22}^{-1}G_{21}G_{11}^{-1} \quad G_{22}^{-1}\right]'$$

where

$$\Omega_T \equiv T\sum \begin{bmatrix} g_{1,t}(\widehat{\theta}_0) \\ g_{2,t}(\widehat{\theta}_0, \widehat{\xi}_0) \end{bmatrix} \left[g_{1,t}(\widehat{\theta}_0)' \quad g_{2,t}(\widehat{\theta}_0, \widehat{\xi}_0)'\right] \to \begin{bmatrix} \mathcal{I}_{FI} & \Omega_{21}' \\ \Omega_{21} & \Omega_{22} \end{bmatrix}$$

References

Borovička, Jaroslav, Lars Peter Hansen, Mark Hendricks, and José A. Scheinkman. 2011. "Risk-Price Dynamics." *Journal of Financial Econometrics* 9 (1):3–65.

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