Online Appendix

This online appendix supports the paper "Twisted Probabilities, Uncertainty, and Prices" by Lars Peter Hansen, Bálint Szőke, Lloyd S. Han, and Thomas J. Sargent.

A Relative entropy neighborhoods reconsidered

We prove the inequality stated in Proposition 3.2. Define the probability measure conditioned on $X_0 = x$ implied by the martingale M^H and construct the product probability measure that includes the time dimension by using the density $\delta \exp(-\delta t)$ over $t \geqslant 0$ for any $\delta > 0$. Call the expectation operator (conditioned on $X_0 = x$) associated with this measure E^H and use it to define the norm

$$\|H\|_H \doteq \left(E^H |H|^2\right)^{1/2}$$

For notational convenience leave the conditioning implicit. Notice that we can express

$$\widehat{\Delta}\left(M^H, 1\right) = \frac{1}{2}\|H\|_H^2 \qquad \text{and} \qquad \widehat{\Delta}\left(M^H, M^{\hat{S}}\right) = \frac{1}{2}\|H - \widehat{S}\|_H^2$$

Define

$$\tau \doteq \min_H \ \delta \int_0^\infty \exp(-\delta\tau) E\left[M_\tau^H \xi(X_\tau) \ \mid \ \mathcal{F}_0\right] d\tau = \min_H \ E^H[\xi(X)].$$

Suppose now that

$$\|\widehat{S}\|_H^2 \leqslant \lambda^2 E^H[\xi(X)] \qquad \text{and} \qquad \|H - \widehat{S}\|_H^2 \leqslant (1-\lambda)^2 \tau.$$

By the Triangle Inequality and the concavity of the square root function

$$\begin{aligned} \|H\|_H \leqslant \|H - \widehat{S}\|_H + \|\widehat{S}\|_H &\leqslant \lambda \left(E^H[\xi(X)]\right)^{1/2} + (1-\lambda)\tau^{1/2} \\ &< \left(\lambda E^H[\xi(X)] + (1-\lambda)\tau\right)^{1/2} \leqslant \left(E^H[\xi(X)]\right)^{1/2} \end{aligned}$$

Consequently, $\varrho\left(M^H; \xi\right) < 0$. $\square$

B Results for the game with a statistician

The following table is a counterpart of Table 1 in section 6.1.4. The only difference is the way the worst-case models are computed. While for Table 1 we use the zero-sum game formulation in section 4, for Table B.1 we use the statistician's game discussed in section 5.

q	$\widetilde{\alpha}_z$	$\widetilde{\kappa}$	$\widetilde{\beta}$	α_c	β	α_z	κ	$\Delta\bar{c}$	m_z	s_z
Baseline										
0.000	0.000	0.014	1.000	0.484	1.000	0.000	0.014	0.000	0.000	0.163
State-dependent $\xi^{[\kappa]}$										
0.100	-0.002	0.010	1.000	0.460	1.005	-0.003	0.013	-0.218	-0.193	0.167
0.200	-0.003	0.010	1.000	0.436	1.005	-0.005	0.013	-0.436	-0.386	0.167
0.100	-0.000	0.005	1.000	0.467	1.028	-0.002	0.010	-0.208	-0.186	0.193
0.200	-0.001	0.005	1.000	0.451	1.028	-0.004	0.010	-0.429	-0.385	0.193
State-dependent $\xi^{[\beta]}$										
0.100	-0.002	0.014	1.157	0.465	1.028	-0.002	0.010	-0.209	-0.185	0.193
0.200	-0.003	0.014	1.199	0.459	1.053	-0.002	0.006	-0.410	-0.366	0.240
0.100	-0.000	0.014	1.164	0.468	1.031	-0.002	0.010	-0.206	-0.184	0.198
0.200	-0.001	0.014	1.208	0.467	1.062	-0.002	0.005	-0.390	-0.352	0.267

Table B.1: Worst-case parameter values implied by the section 5 formulation when ξ is defined by (30). The change in the long run consumption growth expectation is denoted by $\Delta\bar{c} \doteq \left(\alpha_c + \frac{\beta\alpha_z}{\kappa}\right) - \left(\widehat{\alpha}_c + \frac{\widehat{\beta}\widehat{\alpha}_z}{\hat{\kappa}}\right)$. Note that $\left(\widehat{\alpha}_c + \frac{\widehat{\beta}\widehat{\alpha}_z}{\hat{\kappa}}\right) = .484$. m_z and s_z denote the unconditional mean and standard deviation of Z under the worst-case model.

C Robust value functions

We provide formulas and discuss methods to compute the value function for the robust control problem in section 6. The state vector is

$$X_t \doteq [\log K_t,\ L_t,\ Z_t - \bar{z}]' \qquad \log K_t \doteq \log\left(K_t^{(1)} + K_t^{(2)}\right) \qquad L_t \doteq \log K_t^{(2)} - \log K_t^{(1)}$$

Define the ratio

$$R_t \doteq \frac{K_t^{(2)}}{K_t^{(1)} + K_t^{(2)}} = \frac{\exp(L_t)}{1 + \exp(L_t)}.$$

The period utility function is

$$\upsilon(X,D) = \delta \log\left((1-R)\left(\mathcal{A}_1 - D^{(1)}\right) + R\left(\mathcal{A}_2 - D^{(2)}\right)\right) + \delta \log K$$

where we used the resource constraint

$$C_t = \left[(1-R_t)\left(\mathcal{A}_1 - D_t^{(1)}\right) + R_t\left(\mathcal{A}_2 - D_t^{(2)}\right)\right] K_t.$$

Denote expected capital growth $E_t\left[dK_t^{(i)}/K_t^{(i)}\right]$ for $i = 1, 2$ as

$$\varphi_i\left(D_t^{(i)}, Z_t\right) \doteq D_t^{(i)} - \frac{\phi_i}{2}\left(D_t^{(i)}\right)^2 + (.01)\left(\widehat{\alpha}_z + \widehat{\beta} Z_t\right)$$

State variables then follow

$$\begin{aligned} d\log K_t &= \left[\varphi_1(1-R_t) + \varphi_2 R_t - \frac{(.01)^2\left|\sigma_1(1-R_t) + \sigma_2 R_t\right|^2}{2}\right] dt + (.01)\left[\sigma_1(1-R_t) + \sigma_2 R_t\right] \cdot dW_t \\ dL_t &= \left[\varphi_2 - \varphi_1 - \frac{(.01)^2}{2}\left(|\sigma_2|^2 - |\sigma_1|^2\right)\right] dt + (.01)\left[\sigma_2 - \sigma_1\right] \cdot dW_t \\ dZ_t &= -\widehat{\kappa}\left(Z_t - \bar{z}\right) dt + \sigma_z \cdot dW_t \end{aligned}$$

Using Ito's lemma, we can derive the following dynamics for R_t:

$$\begin{aligned} dR_t = R_t(1-R_t)\left[\varphi_2 - \varphi_1 + (.01)^2\left(|\sigma_1|^2(1-R_t) - |\sigma_2|^2 R_t + \sigma_1'\sigma_2(2R_t - 1)\right)\right] dt + \\ + R_t(1-R_t)(.01)\left[\sigma_2 - \sigma_1\right] \cdot dW_t. \end{aligned}$$

Let σ denote the stacked volatility matrix

$$\sigma(X_t) \doteq \begin{bmatrix} (.01)\left(\sigma_1'(1-R_t) + \sigma_2' R_t\right) \\ (.01)[\sigma_2 - \sigma_1]' \\ \sigma_z' \end{bmatrix}.$$

We seek a value function $V(X) = \log K + \nu(L, Z)$ that solves the HJB equation

$$0 = \max_{d^{(1)}, d^{(2)}} \min_h \; \delta \log\left((1-r)\left(\mathcal{A}_1 - d^{(1)}\right) + r\left(\mathcal{A}_2 - d^{(2)}\right)\right) - \delta\nu(l,z) + \frac{\ell}{2}\left[|h|^2 - \xi(z)\right]$$

$$+\left[\varphi_1(1-r)+\varphi_2 r-\frac{(.01)^2\left[\sigma_1(1-r)+\sigma_2 r\right]^2}{2}+(.01)[(1-r)\sigma_1+r\sigma_2]\cdot h\right]$$
$$+\nu_l(l,z)\left[\varphi_2-\varphi_1-\frac{(.01)^2}{2}\left(|\sigma_2|^2-|\sigma_1|^2\right)+(.01)[\sigma_2-\sigma_1]\cdot h\right]$$
$$+\nu_z(l,z)\left[-\widehat{\kappa}(z-\bar{z})+\sigma_z\cdot h\right]+\frac{1}{2}\mathrm{tr}\left(V_{xx}\sigma\sigma'\right) \tag{C.1}$$

where

$$\mathrm{tr}\left(V_{xx}\sigma\sigma'\right)=(.01)^2|\sigma_2-\sigma_1|^2\nu_{ll}(l,z)+2(.01)\left([\sigma_2-\sigma_1]\cdot\sigma_z\right)\nu_{lz}(l,z)+|\sigma_z|^2\nu_{zz}(l,z).$$

We assume that a Bellman-Isaacs condition holds so that first-order conditions can be stacked

$$\frac{\delta(1-r)}{(1-r)(\mathcal{A}_1-d^{(1)}(l,z))+r(\mathcal{A}_2-d^{(2)}(l,z))}=\left(1-\phi_1 d^{(1)}(l,z)\right)\left[1-r-\nu_l(l,z)\right] \tag{C.2}$$
$$\frac{\delta r}{(1-r)(\mathcal{A}_1-d^{(1)}(l,z))+r(\mathcal{A}_2-d^{(2)}(l,z))}=\left(1-\phi_2 d^{(2)}(l,z)\right)\left[r+\nu_l(l,z)\right] \tag{C.3}$$
$$h(l,z,\ell^*)=-\frac{1}{\ell^*}\sigma'(r)\begin{bmatrix}1\\ \nu_l(l,z)\\ \nu_z(l,z)\end{bmatrix}. \tag{C.4}$$

These equations determine optimal investment-capital ratios $d^{(1)}(l,z)$ and $d^{(2)}(l,z)$, and also the worst-case drift distortion $h(l,z)$. Here ℓ^* is the multiplier that makes the minimizing agent's constraint bind for a given initial (l_0,z_0):

$$\ell^*(l_0,z_0)=\arg\max_{\ell}\quad \nu(l_0,z_0,\ell).$$

C.1 Single capital stock

The "boundaries" $r=0$ and $r=1$ can be described in terms of two single-capital economies. The HJB equation becomes

$$0=\max_{d^{(i)}}\min_{h}\ \delta\log(\mathcal{A}_i-d^{(i)})-\delta\nu(z)+\frac{\ell}{2}\left[|h|^2-\xi(z)\right]+ \tag{C.5}$$
$$+\left[d^{(i)}-\frac{\phi_i}{2}[d^{(i)}]^2+(.01)\left(\widehat{\alpha}_i+\widehat{\beta}_i\bar{z}+\widehat{\beta}_i(z-\bar{z})\right)-\frac{(.01)^2\,|\sigma_i|^2}{2}+(.01)\sigma_i\cdot h\right]+$$

$$+\nu_z(z)\left[-\widehat{\kappa}(z-\bar{z})+\sigma_z\cdot h\right]+\frac{1}{2}\text{tr}\left(V_{xx}\sigma\sigma'\right)$$

with $i=1$ when $r=0$ and $i=2$ when $r=1$. The optimal choice $d^{(i)}$ is given by (C.2), namely,

$$d^*=\frac{1}{2}\left[\mathcal{A}_i+\frac{1}{\phi_i}-\sqrt{\left(\frac{1}{\phi_i}-\mathcal{A}_i\right)^2+\frac{4\delta}{\phi_i}}\right]$$

With $\xi(z)=\xi_0+2\xi_1(z-\bar{z})+\xi_2(z-\bar{z})^2$, the value function $\nu(z)$ is quadratic

$$\nu(z,\ell)=\frac{1}{2}\left[\nu_0(\ell)+2\nu_1(\ell)(z-\bar{z})+\nu_2(\ell)(z-\bar{z})^2\right];$$

$\nu_0(\ell)$, $\nu_1(\ell)$, $\nu_2(\ell)$ can be obtained by plugging optimal policies into HJB equation (C.5) and matching coefficients

$$\nu_2(\ell)=-\ell\left[\frac{\delta+2\widehat{\kappa}-\sqrt{(\delta+2\widehat{\kappa})^2-4|\sigma_z|^2\xi_2}}{2|\sigma_z|^2}\right]\doteq-\ell\omega_2$$

$$\nu_1(\ell)=\frac{-\ell\xi_1+(.01)\widehat{\beta}_i+(.01)\omega_2(\sigma_i\cdot\sigma_z)}{\delta+\widehat{\kappa}-\omega_2|\sigma_z|^2}$$

$$\delta\nu_0(\ell)=\frac{1}{\delta}\left\{2\left[\delta\log(\mathcal{A}_i-d^*)+d^*-\frac{\phi_i}{2}(d^*)^2+(.01)\left(\widehat{\alpha}_i+\widehat{\beta}_i\bar{z}\right)\right]\right.$$
$$\left.-(.01)^2|\sigma_i|^2-\ell\xi_0+\nu_2(\ell)|\sigma_z|^2-\frac{1}{\ell}|(.01)\sigma_i+\sigma_z\nu_1(\ell)|^2\right\}.$$

For a given initial z_0, the multiplier ℓ^* satisfies

$$\ell^*(z_0)=\arg\max_{\ell}\ \frac{1}{2}\left[\nu_0(\ell)+2\nu_1(\ell)(z_0-\bar{z})+\nu_2(\ell)(z_0-\bar{z})^2\right].$$

C.2 Numerical Method

For the value function $\nu(l,z)$ in the two-capital-stock problem, we solve HJB equation (C.1) numerically using the finite difference method with implicit upwind scheme described by Candler (2001) and the Online Appendix of Achdou et al. (2017). We construct a two-dimensional grid for l and z, so that $(l,z)\in[-l^\star,l^\star]\times[-z^\star,z^\star]$. We set $l^\star=20$ and $z^\star=1.2$.

For a given ℓ, we use the finite difference method to derive the function $\nu(l_i, z_j; \ell)$ on the grid. To initialize iterations, we exploit that the value functions at the $r = 0$ ($l = -\infty$) and $r = 1$ ($l = \infty$) boundaries are known (with z as their only argument); we extend these function to the whole grid by linearly interpolating between them (over l). Using formulas from Appendix C.1, we can also derive $\ell^*(-\infty, z_0)$ and $\ell^*(\infty, z_0)$. We find the optimal multiplier $\ell^*(l_0, z_0)$ *at a given* (l_0, z_0) by maximizing $\nu(l_0, z_0; \ell)$ with respect to ℓ. A good initial guess for the optimizer is $\ell^0 = \frac{\ell^*(-\infty, z_0) + \ell^*(\infty, z_0)}{2}$.

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2017. Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach. Working Paper 23732, National Bureau of Economic Research.

Candler, Graham. 2001. Finite-Difference Methods for Continuous-Time Dynamic Programming. In *Computational Methods for the Study of Dynamic Economies*, edited by Ramon Marimon and Andrew Scott. Cambridge, England: Cambridge University Press.