Estimating robustness
– Online Appendix –

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1 Results for the restricted sample

This appendix presents results for the period 1952:Q2-2005:Q4, which coincides with the sample period used by Piazzesi and Schneider (2007). To facilite comparison with their analysis, I also include estimates for the case when the tilting function ξ is restricted to be constant, which is interpretable as recursive utility or the robust control model of Hansen and Sargent (2001) with unstructured uncertainty (see Section ??).

κ^D		σ^D		β_0^D	α^D	
0.568	-0.066	0.129	-0.038	0.533	0.434	0.000
(0.139)	(0.043)	(0.030)	(0.029)	-	(0.021)	-
0.225	0.987	-0.002	0.167	0.920	-0.089	0.300
(0.104)	(0.030)	(0.024)	(0.021)	-	(0.021)	(0.016)

Table 1: Maximum Likelihood estimates and asymptotic standard errors (in parentheses) for the baseline model (??)-(??) when the sample is restricted to the period 1952:Q2-2005:Q4. The likelihood is initialized at the stationary distribution of X. The column for β_0^D shows the sample averages. The matrix α^D is normalized to be lower triangular.

Table 1 contains maximum likelihood estimates of the baseline model parameters. These values are more or less in line with the findings of Piazzesi and Schneider (2007). Apparently, in the restricted sample consumption growth appears to be less persistent, while inflation is even more persistent than in the extended sample used in the main text. Figure 1 displays the corresponding autrocorrelation functions along with the sample analogues calculated using the restricted sample. As hinted before, the main difference between the two samples can be spotted on the top right panel: for the restricted sample, high inflation seems to be a good predictor of low future consumption growth, at least on a 1-2 year horizon.

As for the parameters of the tilting function, Table 2 contains the non-linear least squares estimates. Assuming the quadratic form (??) leads to parameter estimates significantly different from the ones in Table ??. The diagonal elements of the Ξ matrix are an order of magnitude larger and the cross term $\bar{\xi}_2$ even changes sign. This suggests that the properties of the baseline model, especially the estimated persistence of consumption growth and inflation can significantly affect the particular values of the

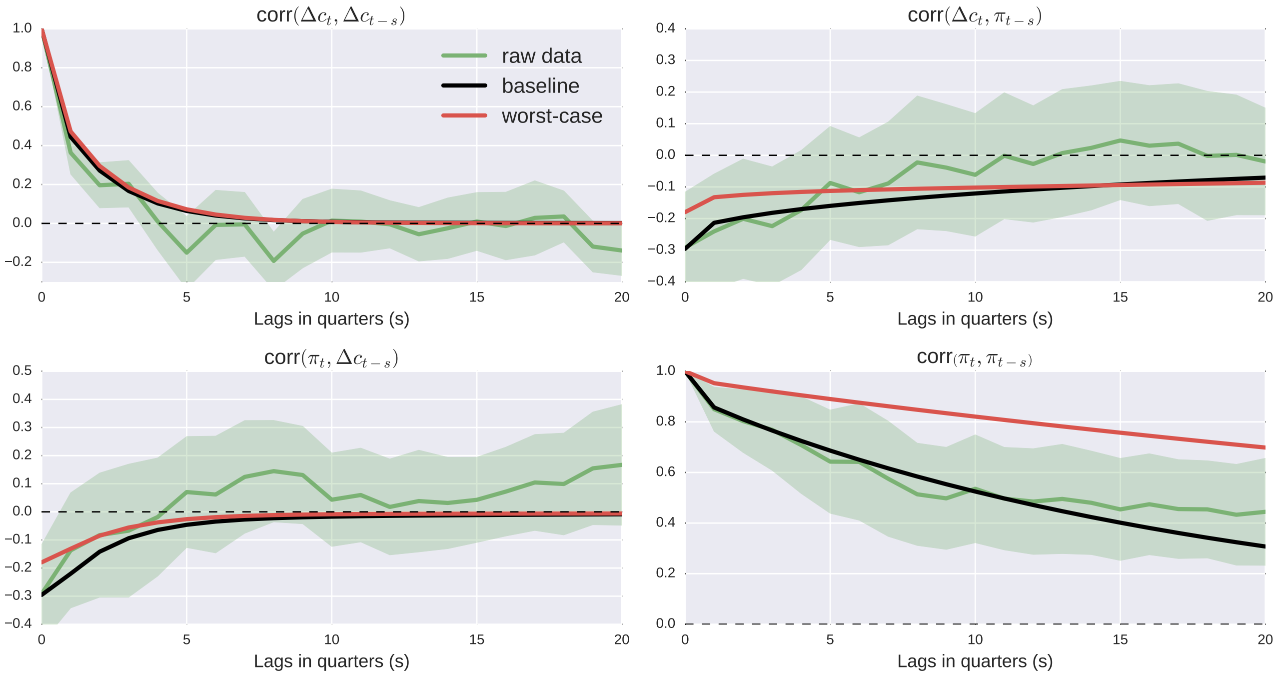

Figure 1: Autocorrelation functions computed from raw data (green solid lines) and from the baseline model (black solid lines) when the sample is restricted to the period 1952:Q2-2005:Q4. Shaded area represents 2×GMM standard error bounds computed with the Newey-West estimator including 4 quarter lags.

tilting function parameters. Nevertheless, these drastic changes in the parameter values do not alter the model's overall prediction about the worst-case distribution. As the red lines on Figure 1 demonstrate, the worst-case model indicates more persistence than the baseline.

The relationship between the red and black lines on the top right panel provides explanations for the changing sign of $\bar{\xi}_2$. The estimated baseline model assigns such a strong forecasting ability to inflation that the worst-case model wants to counteract it, at least on the short horizon (up to 3-year). Because this forecasting channel is shaped mainly by the parameter $\bar{\xi}_2$, the estimated sign is flipped.

	$\bar{\xi}_0$		$\bar{\xi}_1$	$\bar{\xi}_2$	$\bar{\xi}_3$
Quadratic tilting	0.0		11,829	4,537	1,740
	(0.02)		(8,125)	(1,031)	(201)
Recursive utility	0.077	$\Rightarrow$	$\gamma = 32$		
	(0.013)				

Table 2: Non-linear least squares estimates and asymptotic GMM standard errors (in the parentheses) for the parameters of ξ when the sample is restricted to the period 1952:Q2-2005:Q4. The standard errors take into account the two-step nature of the estimator. γ denotes the risk aversion parameter of recursive utility and is calculated from $\bar{\xi}_0$ using the formula in Section **??**.

As for the special case of recursive utility, the bottom panel of Table 2 shows that in the restricted sample, recursive utility does have a role. The best fit is reached when the risk aversion parameter is around 30, which is in the ballpark of the estimate of Piazzesi and Schneider (2007). Nevertheless, the right panel of Figure 2 reveals that even though recursive utility is able to generate an upward sloping average nominal yield curve in the restricted sample, the predicted variation is quickly decreasing with the horizon, which is at odds with the data. Again, we need state dependence in ξ in order to break the expectations hypothesis and induce time-variation in risk premia.

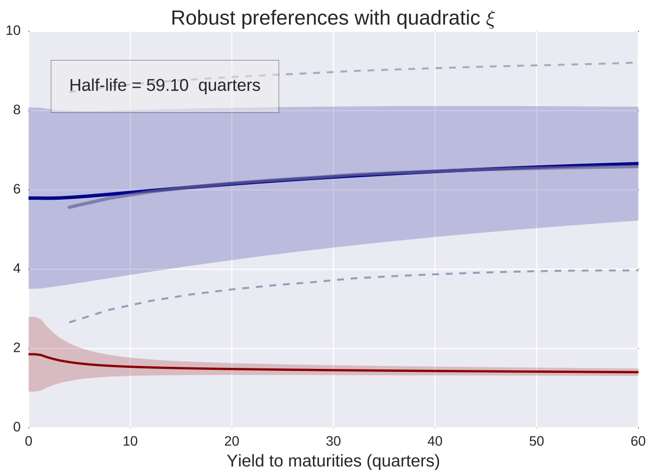

Figure 2: Model implied stationary distributions of the nominal (blue) and real (red) yield curves and the corresponding sample moments when the sample is restricted to the period 1952:Q2-2005:Q4. Shaded areas represent one standard deviation bands around the means (solid lines). Grey solid lines show the sample average, the dashed lines are one standard deviation bands using the sample standard deviations. The left panel is for the model with a constant ξ (recursive utility). The right panel shows the case for expected utiliy with logarithmic preferences. The boxes contain associated half-lives.

2 Yield curve

The price of a zero-coupon bond with maturity τ is

$$P_t^{(\tau)}(x) = \widetilde{\mathbb{E}}\left[\exp(-\delta\tau)\frac{(CP)_t}{(CP)_{t+\tau}} \mid X_t = x\right] = \widetilde{\mathbb{E}}\left[\exp(-\delta\tau)\frac{(CP)_0}{(CP)_\tau} \mid X_0 = x\right]$$

The second equality follows from the fact that since both C and P are multiplicative functionals, their product is also a multiplicative functional. This implies that the equilibrium yield curve is time-invariant and it hinges only on the current value of X. Using the given law of motions for C and P, we can write

$$\begin{aligned} P^{(\tau)}(x) &= \widetilde{\mathbb{E}}\left[\exp\left(-\delta\tau - \int_0^\tau d\log C_s - \int_0^\tau d\log P_s\right) \mid X_0 = x\right] = \\ &= \widetilde{\mathbb{E}}\left[\exp\left(\int_0^\tau \left[-\delta - \iota_2\cdot\beta_0 - \iota_2^T\beta_1 X_s\right] ds + \int_0^\tau -\iota_2^T\alpha dW_s\right) \mid X_0 = x\right] \end{aligned}$$

Let the exponential inside the expectation operator be M_t, and translate everything into the notation of BHHS (section XXX), to derive ODEs for $a(\tau)$ and $b(\tau)$ in the expression of the the expectation of a multiplicative functional with affine dynamics, i.e. $\widetilde{\mathbb{E}}[M_t \mid X_0 = x] = \exp(a(t) + b(t)x)$. The dictionary between the notations is as follows

$$\beta(x) = \bar{\beta}_0 + \bar{\beta}_1 \cdot (x - \bar{x}) = \left[-\delta - \iota_2 \cdot \widetilde{\beta}_0 - \iota_2^T \widetilde{\beta}_1 \bar{x}\right] + \left[-\widetilde{\beta}_1^T \iota_2\right] \cdot (x - \bar{x}) \qquad \bar{\alpha} = -\alpha^T \iota_2$$

where the Markov state vector has the following drift and volatility terms

$$\mu(x) = \bar{\mu}_{11}(x - \bar{x}) = \widetilde{\phi} - \widetilde{\kappa} x = (-\widetilde{\kappa}) \left[x - \widetilde{\kappa}^{-1} \widetilde{\phi}\right] \qquad \bar{\sigma}_1 = \sigma$$

where $\widetilde{\kappa} = \kappa - \sigma \eta_1$. Then the ODEs are as follows

$$\begin{aligned} \frac{d}{dt} b(t)^T &= \bar{\beta}_1^T + b(t)^T \bar{\mu}_{11} \\ \frac{d}{dt} a(t) &= \bar{\beta}_0 - \left[\bar{\beta}_1^T + b(t)^T \bar{\mu}_{11}\right] \bar{x} + \frac{1}{2}\left(\bar{\alpha}^T \bar{\alpha} + 2\bar{\alpha}^T \bar{\sigma}_1^T b(t) + b(t)^T \bar{\sigma}_1 \bar{\sigma}_1^T b(t)\right) \end{aligned}$$

using the paper's notation

$$\begin{aligned} \frac{d}{dt} b(t)^T &= -\iota_2^T \beta_1 + b(t)^T (-\widetilde{\kappa}) \\ \frac{d}{dt} a(t) &= -\delta - \iota_2 \cdot \widetilde{\beta}_0 - b(t)^T (-\widetilde{\kappa}) \bar{x} + \frac{1}{2}\left(\iota_2^T \alpha \alpha^T \iota_2 - 2\iota_2^T \alpha \sigma^T b(t) + b(t)^T \sigma \sigma^T b(t)\right) \end{aligned}$$

Let κ^D denote the coefficient matrix coming from the estimation of the discretized baseline state space, i.e.

$$\kappa^D \equiv I - \kappa \qquad \qquad \widetilde{\kappa} = I - \kappa^D - \sigma \eta_1$$

Then $(-\widetilde{\kappa})\widetilde{\kappa}^{-1}\phi = -\phi$. Using this, the ODEs become

$$\begin{aligned} \frac{d}{dt} b(t)^T &= -\iota_2^T \widetilde{\beta}_1 + b(t)^T (\kappa^D - I + \sigma \eta_1) = -\widetilde{\rho}_0 + b(t)^T (\kappa^D + \sigma \eta_1 - I) \\ \frac{d}{dt} a(t) &= -\left[\delta + \iota_2 \cdot \widetilde{\beta}_0 - \frac{|\alpha^T \iota_2|^2}{2}\right] + b(t)^T \widetilde{\phi} - \iota_2^T \alpha \sigma^T b(t) + \frac{1}{2} b(t)^T \sigma \sigma^T b(t) \end{aligned}$$

The finite (forward) difference version of which is

$$\begin{aligned} b(t+\Delta)^T &= b(t)^T (1 - \Delta) - \Delta \widetilde{\rho}_0 + \Delta b(t)^T (\kappa^D + \sigma \eta_1) \\ a(t+\Delta) &= a(t) + \Delta \left[-\widetilde{\rho}_1 + b(t)^T \phi - \iota_2^T \alpha \sigma^T b(t) + \frac{1}{2} b(t)^T \sigma \sigma^T b(t)\right] \end{aligned}$$

with $a(0) = 0$ and $b(0) = 0$.

3 Chernoff entropy

Relative to the benchmark model we derive alternative models through martingales $\frac{dZ^H}{Z_t^H} = H_t \cdot dW_t$ and parametrize them by their exposure H_t. Since H_t takes the form $H_t = \eta(X_t)$, the alternative models are Markovians.

Given the process H, Chernoff-entropy is defined as

$$\chi(Z_t^H, x) = - \inf_{0 \leqslant \mathbf{r} \leqslant 1} \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}\left[(Z_t^H)^{\mathbf{r}} | X_0 = x\right]$$

Both Z_t^H and $\left(Z_t^H\right)^{\mathbf{r}}$ are multiplicative functionals, so apart from the inf over $\mathbf{r}$, χ is equal to the long-term growth rate (provided it is well-defined) of the implied multiplicative semigroup. Hansen-Scheinkman tells us how to calculate this growth rate using the principal eigenvalue problem

$$\mathbb{E}\left[\left(Z_t^H\right)^{\mathbf{r}} e(X_t) | X_0 = x\right] = \exp(-\psi t) e(x)$$

for all t where e is strictly positive. The local counterpart is

$$[\mathbb{G}e](x) = \lim_{\tau \downarrow 0} \frac{\mathbb{E}\left[\left(Z_\tau^H\right)^{\mathbf{r}} e(X_\tau) | X_0 = x\right] - e(x)}{\tau} = -\psi e(x)$$

so we need the drift of $\left(Z_t^H\right)^{\mathbf{r}} e(X_t)$. Ito's lemma implies

$$\begin{aligned}
d\left(Z_t^H\right)^{\mathbf{r}} &= \frac{\mathbf{r}(\mathbf{r}-1)}{2}\left(Z_t^H\right)^{\mathbf{r}} |H_t|^2 dt + \mathbf{r}\left(Z_t^H\right)^{\mathbf{r}} H_t \cdot dW_t \\
de(X_t) &= \left[\nabla e(X_t)^T(\phi - \kappa X_t) + \frac{1}{2}\mathrm{tr}\left(\sigma^T e_{xx}(X_t)\sigma\right)\right] dt + \nabla e(X_t)^T \sigma dW_t \\
d\left[\left(Z_t^H\right)^{\mathbf{r}} e(X_t)\right] &= \frac{\mathbf{r}(\mathbf{r}-1)}{2}\left(Z_t^H\right)^{\mathbf{r}} |H_t|^2 e(X_t) + \left[\nabla e(X_t)^T(\phi - \kappa X_t) + \frac{1}{2}\mathrm{tr}\left(\sigma^T e_{xx}(X_t)\sigma\right)\right]\left(Z_t^h\right)^{\mathbf{r}} + \\
&\quad + \mathbf{r}\left(Z_t^h\right)^{\mathbf{r}} \nabla e(X_t)^T \sigma H_t
\end{aligned}$$

evaluating it at $t = 0$ yields

$$\begin{aligned}
[\mathbb{G}e](x) &= \frac{\mathbf{r}(\mathbf{r}-1)}{2}|H_t|^2 e(x) + \left[\nabla e(X_t)^T(\phi - \kappa X_t) + \frac{1}{2}\mathrm{tr}\left(\sigma^T e_{xx}(X_t)\sigma\right)\right] + \mathbf{r}\nabla e(X_t)^T \sigma H_t \\
-\psi &= \frac{\mathbf{r}(\mathbf{r}-1)}{2}|H_t|^2 + \nabla \log e(x)^T(\underbrace{\phi + \mathbf{r}\sigma\eta_0}_{\equiv\tilde{\phi}} - \underbrace{\kappa - \mathbf{r}\sigma\eta_1]}_{\equiv\tilde{\kappa}} x) + \frac{1}{2}\mathrm{tr}\left(\sigma^T \frac{1}{e(x)} e_{xx}(x)\sigma\right)
\end{aligned}$$

If $h(X_t)$ is affine in X_t, then guess and verify shows that the eigenfunction is exponential of a quadratic function of X_t. Use the form $\log e(x) = \lambda_0 + \lambda_1 \cdot x + x^T \lambda_2 x$, so

$$\begin{aligned}
\nabla \log e(x)^T &= \lambda_1^T + x^T\left(\lambda_2^T + \lambda_2\right) \\
\frac{1}{e(x)} e_{xx}(x) &= \left(\lambda_2^T + \lambda_2\right) + \left[\lambda_1 + \left(\lambda_2 + \lambda_2^T\right) x\right]\left[\lambda_1^T + x^T\left(\lambda_2^T + \lambda_2\right)\right]
\end{aligned}$$

and

$$\frac{1}{2}\mathrm{tr}\left(\sigma^T \frac{1}{e(x)} e_{xx}(x)\sigma\right) = \frac{1}{2}\mathrm{tr}\left(\sigma^T \left(\lambda_2^T + \lambda_2\right)\sigma\right) + \frac{1}{2}\mathrm{tr}\left(\sigma^T \lambda_1 \lambda_1^T \sigma\right) + \frac{1}{2}\mathrm{tr}\left(\sigma^T \left(\lambda_1 x^T (\lambda_2^T + \lambda_2)\right)\sigma\right) +$$
$$+ \frac{1}{2}\mathrm{tr}\left(\sigma^T \left((\lambda_2^T + \lambda_2)\right) x \lambda_1^T \sigma\right) + \frac{1}{2}\mathrm{tr}\left(\sigma^T (\lambda_2^T + \lambda_2) x x^T (\lambda_2^T + \lambda_2)\sigma\right)$$
$$= \frac{1}{2}\mathrm{tr}\left(\sigma^T \left(\lambda_2^T + \lambda_2 + \lambda_1\lambda_1^T\right)\sigma\right) + x^T \left(\lambda_2^T + \lambda_2\right)\sigma\sigma^T \lambda_1 +$$
$$+ \frac{1}{2} x^T (\lambda_2^T + \lambda_2)\sigma\sigma^T (\lambda_2^T + \lambda_2) x$$

and the ODE becomes

$$-\psi = \frac{\mathbf{r}(\mathbf{r}-1)}{2}|H_t|^2 + \left(\lambda_1^T + x^T(\lambda_2^T + \lambda_2)\right)(\phi - \kappa x + \mathbf{r}\sigma H_t) + \frac{1}{2}x^T(\lambda_2^T + \lambda_2)\sigma\sigma^T(\lambda_2^T + \lambda_2)x +$$
$$+ \frac{1}{2}\mathrm{tr}\left(\sigma^T \left(\lambda_2^T + \lambda_2 + \lambda_1\lambda_1^T\right)\sigma\right) + x^T \left(\lambda_2^T + \lambda_2\right)\sigma\sigma^T\lambda_1$$

Matching coefficients for x^2 yields

$$0 = \frac{\mathbf{r}(\mathbf{r}-1)}{2} x^T \eta_1^T \eta_1 x - x^T(\lambda_2^T \tilde{\kappa} + \lambda_2\tilde{\kappa})x + 2x^T\lambda_2\sigma\sigma^T\lambda_2 x$$
$$0 = \frac{\mathbf{r}(\mathbf{r}-1)}{2}\eta_1^T\eta_1 - \tilde{\kappa}^T\lambda_2 - \lambda_2\tilde{\kappa} + 2\lambda_2\sigma\sigma^T\lambda_2$$

which is a continuous time algebraic Riccati equation of the form

$$A^T\lambda_2 + \lambda_2 A - \lambda_2 B R^{-1} B^T \lambda_2 + Q = 0$$

where

$$A = \tilde{\kappa} = \kappa - \mathbf{r}\sigma\eta_1 \qquad B = \sqrt{2}\sigma \qquad R = -I_K$$

Matching coefficients for x^T

$$0 = \mathbf{r}(\mathbf{r}-1)\eta_1^T\eta_0 + (\lambda_2^T + \lambda_2)\widetilde{\phi} - \left(\tilde{\kappa} - (\lambda_2^T + \lambda_2)\sigma\sigma^T\right)\lambda_1$$
$$\lambda_1 = \left(\tilde{\kappa} - (\lambda_2^T + \lambda_2)\sigma\sigma^T\right)^{-1}\left[\mathbf{r}(\mathbf{r}-1)\eta_1^T\eta_0 + (\lambda_2^T + \lambda_2)\widetilde{\phi}\right]$$

and the scalar terms can be solved for ψ

$$-\psi(\mathbf{r}) = \frac{\mathbf{r}(\mathbf{r}-1)}{2}\eta_0 \cdot \eta_0 + \lambda_1^T\widetilde{\phi} + \frac{1}{2}\mathrm{tr}\left(\left(\lambda_2^T + \lambda_2 + \lambda_1\lambda_1^T\right)\sigma\sigma^T\right)$$

Maximizing $\psi(\mathbf{r})$ over $[0, 1]$ w.r.t $\mathbf{r}$ gives the Chernoff entropy associated with H. The half-life then can be calculated as

$$HL = \frac{\log(2)}{\max_{\mathbf{r}\in[0,1]} \psi(\mathbf{r})}$$

4 Discrete sampling from the continuous path

The state vector X_t follows a multivariate Ornstein-Uhlenbeck process, so it has a conditional normal distribution for all t. Define the unconditional mean as $\mu \equiv \kappa^{-1}\phi$. The SDE can be written as

$$dX_t = -\kappa\left(X_t - \mu\right)dt + \sigma dW_t$$

For an arbitrary $\tau \geqslant 0$, the distribution of $X_{t+\tau}$ given X_t follows

$$X_{t+\tau}|X_t \sim \mathcal{N}\left(\bar{x}_{t+\tau},\ \ \Sigma_\tau\right)$$

with

$$\bar{x}_{t+\tau} = (I - \exp(-\kappa\tau))\mu + \exp(-\kappa\tau)X_t \qquad \text{vec}(\Sigma_\tau) = \left(\kappa \oplus \kappa\right)^{-1}\left(I - \exp(-(\kappa\oplus\kappa)\tau)\right)\text{vec}(\sigma\sigma^T)$$

At the same time, the additive functional vector $d\ln Y_t$ follows

$$d\log Y_t = \left(\beta_0 + \beta_1 X_t\right)dt + \alpha dW_t$$

hence

$$\begin{aligned}\log Y_{t+\tau} - \log Y_t = \beta_0\tau + \beta_1\int_t^{t+\tau} X_s ds + \int_t^{t+\tau}\alpha dW_t = \left(\beta_0 + \beta_1\mu\right)\tau + \int_t^{t+\tau}\beta_1\exp\left(-\kappa s\right)\left(X_t - \mu\right)ds + \\ + \int_t^{t+\tau}\beta_1\left[\int_t^s \exp\left(-\kappa(s-u)\right)\sigma dW_u\right]ds + \int_t^{t+\tau}\alpha dW_s\end{aligned}$$

This implies that the conditional distribution of $\log Y_{t+\tau} - \log Y_t$ is normal with **mean**

$$\begin{aligned}\mathbb{E}\left[\log Y_{t+\tau} - \log Y_t \mid X_t = x\right] &= \left(\beta_0 + \beta_1\mu\right)\tau + \beta_1\kappa^{-1}\left(I - \exp(-\kappa\tau)\right)\left[x - \mu\right] = \\ &= \underbrace{\beta_0\tau + \beta_1\left(I - \kappa^{-1} + \kappa^{-1}\exp(-\kappa\tau)\right)\mu\tau}_{\equiv\beta_{0D}} + \underbrace{\beta_1\kappa^{-1}\left(I - \exp(-\kappa\tau)\right)}_{\equiv\beta_{1D}}x\end{aligned}$$

The **covariance matrix** of $\log Y_{t+\tau} - \log Y_t$ given X_t comes from the exposure to $\int_t^{t+\tau} dW_s$

$$\beta_1\int_t^{t+\tau}\left[\int_t^s \exp(-\kappa(s-u))\sigma dW_u\right]ds + \int_t^{t+\tau}\alpha dW_s$$

The first term can be rewritten as

$$\begin{aligned}\beta_1\int_t^{t+\tau}\left[\int_t^s \exp(-\kappa(s-u))\sigma dW_u\right]ds &= \beta_1\int_t^{t+\tau}\left[\int_u^{t+\tau}\exp(-\kappa(s-u))\sigma ds\right]dW_u = \\ &= \beta_1\int_t^{t+\tau}\left(I - \exp(\kappa(u - [t+\tau]))\right)\kappa^{-1}\sigma dW_u\end{aligned}$$

Therefore the covariance matrix is determined by the term

$$\int_t^{t+\tau} \left[\beta_1 \kappa^{-1}\sigma + \alpha - \beta_1 \exp(\kappa(u - [t+\tau]))\kappa^{-1}\sigma\right] dW_u$$

Using this with the Ito isometry we arrive at the covariance matrix

$$\begin{aligned}\left(\beta_1\kappa^{-1}\sigma + \alpha\right)\left(\alpha^T + \sigma^T(\kappa^T)^{-1}\beta_1^T\right)\tau - \int_t^{t+\tau} \left[\alpha + \beta_1\kappa^{-1}\sigma\right]\sigma^T(\kappa^T)^{-1}\exp(\kappa^T(u-[t+\tau]))\beta_1^T du \\ - \int_t^{t+\tau} \beta_1 \exp(\kappa(u-[t+\tau]))\kappa^{-1}\sigma\left[\alpha^T + \sigma^T(\kappa^T)^{-1}\beta_1^T\right] du \\ + \int_t^{t+\tau} \beta_1 \exp(\kappa(u-[t+\tau]))\kappa^{-1}\sigma\sigma^T(\kappa^T)^{-1}\exp(\kappa^T(u-[t+\tau]))\beta_1^T du\end{aligned}$$

the second and third term can be written as, respectively

$$\begin{aligned}\left[\alpha + \beta_1\kappa^{-1}\sigma\right]\sigma^T(\kappa^T)^{-1}\left(I - \exp(-\kappa^T\tau)\right)(\kappa^T)^{-1}\beta_1^T \quad \text{and} \\ \beta_1\kappa^{-1}\left(I - \exp(-\kappa\tau)\right)\kappa^{-1}\sigma\left[\alpha^T + \sigma^T(\kappa^T)^{-1}\beta_1^T\right]\end{aligned}$$

while the fourth term is of the form (without β_1 in the front and the end)

$$T4(\tau) = \left[\int_0^\tau \exp(-\kappa s) Q_c \exp(-\kappa^T s) ds\right]$$

where $Q_c = \kappa^{-1}\sigma\sigma^T(\kappa^T)^{-1}$ is positive semi-definite symmetric matrix. Either Van Loan (1978) algorithm or use the formula

$$\text{vec}(T4(\tau)) = (\kappa \oplus \kappa)^{-1}\left(I - \exp(-(\kappa\oplus\kappa)\tau)\right)\text{vec}(Q_c)$$

5 Relationship between robust control and recursive utility

Recursive utility model with IRS= 1: See Section 5 in Borovicka et al (2009). The stochastic discount factor is multiplicative, the last term is also a martingale, but the exposure of this martingale is not state dependent. For the recursive utility case

$$S_t = \exp\left(-\delta t - \iota \cdot \beta_0 t - \iota^T \beta_1 \int_0^t X_s ds - \frac{|\tilde{\alpha}|^2}{2} t - [\iota^T \alpha - \tilde{\alpha}^T] W_t\right)$$

with $\tilde{\alpha} = (1-\gamma)[\alpha^T e_1 + \sigma^T \bar{v}_1] = (1-\gamma)[\alpha^T e_1 + \sigma^T \nabla_x W]$. Where W is the linear value function with marginal value

$$\nabla_x W = \bar{v}_1 = \left[\delta I_2 - (-\kappa)^T\right]^{-1}\beta_1^T e_1$$

Using Ito's lemma we can also derive

$$\frac{dS_t}{S_t} = -\left(\delta + \iota \cdot \beta_0 + \iota^T \beta_1 X_t - \frac{|\iota^T \alpha|^2}{2} + (\alpha^T \iota) \cdot \tilde{\alpha}\right) dt - [\alpha^T \iota - \tilde{\alpha}] \cdot dW_t$$

Robust preference model: The stochastic discount factor is multiplicative, the last term is also a martingale with exposure $\eta(X)$.

$$S_t = \exp\left(-\delta t - \log(CP)_t - \log(CP)_0 - \frac{|\eta(X_t)|^2}{2} t - [\iota\alpha - \eta(X_t)] W_t\right)$$

with

$$\eta(X) = -\frac{1}{\theta}\left[\alpha^T e_1 + \sigma^T 2(\bar{v}_{1,1}\theta + \bar{v}_{1,0}) + \sigma^T 2\bar{v}_2 \theta X_t\right] \equiv \eta_0 + \eta_1 X_t$$

Using Ito's lemma

$$\frac{dS_t^N}{S_t^N} = -\left(\delta + \iota \cdot \beta_0 + \iota^T \beta_1 X_t - \frac{|\iota^T \alpha|^2}{2} + (\alpha^T \iota) \cdot \eta(X_t)\right) dt - \left(\alpha^T \iota - H_t^*\right) \cdot dW_t$$

Special case: $\xi_1 = 0$ and $\xi_2 = 0$. Then, $\bar{v}_2 = 0$ and $\bar{v}_{1,1} = 0$ and

$$\bar{v}_{1,0} = \frac{1}{2}\left[\delta I_2 - (-\kappa)^T\right]^{-1} \beta_1^T e_1$$

$$\theta^* = \sqrt{\frac{\bar{v}_{0,-1}}{\bar{v}_{0,1}}} = \sqrt{\frac{\left|\alpha^T e_1 + 2\sigma^T \bar{v}_{1,0}\right|^2}{\xi_0}} = \sqrt{\frac{1}{\xi_0}}\left|\alpha^T e_1 + 2\sigma^T \bar{v}_{1,0}\right|$$

Observational equivalence when

$$(\gamma - 1) = \frac{1}{\theta^*(\xi_0)} = \sqrt{\frac{\xi_0}{\left|\alpha^T e_1 + 2\sigma^T \bar{v}_{1,0}\right|^2}}$$

References

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Piazzesi, Monika and Martin Schneider. 2007. “Equilibrium Yield Curves.” In *NBER Macroeconomics Annual 2006, Volume 21*, NBER Chapters. National Bureau of Economic Research, Inc, 389–472.