Costs of Financing US Federal Debt: 1791-1933*

George Hall[†] Jonathan Payne[‡] Thomas J. Sargent[§] Bálint Szőke[¶]

August 26, 2021

Abstract

We use computational Bayesian methods to estimate parameters of a statistical model of gold, greenback, and real yield curves for US federal debt from 1791 to 1933. Posterior probability coverage intervals indicate more uncertainty about yields during periods in which data are especially sparse. We detect substantial discrepancies between our approximate yield curves and standard historical series on yields on US federal debt, especially during War of 1812 and Civil War surges in government expenditures that were accompanied by units of account ambiguities. We use our approximate yield curves to describe how long it took to achieve Alexander Hamilton's goal of reducing default risk premia in US yields by building a reputation for servicing debts as promised. We infer that during the Civil War suspension of convertibility of greenback dollars into gold dollars, US creditors anticipated a rapid post war return to convertibility at par, but that after the war they anticipated a slower return.

JEL classification: E31, E43, G12, N21, N41

Key words: Big data, default premia, yield curve, units of account, gold standard, government debt, Hamiltonian Monte Carlo, Julia, DynamicHMC.jl, pricing errors, specification analysis.

*We thank Clemens Lehner for outstanding research assistance and Min Wei and audiences at the Minnesota Workshop in Macroeconomic Theory and a University of Sydney seminar for suggestions. The views expressed here are those of the authors and do not necessarily represent the views of the Federal Reserve Board or its staff.

[†]Brandeis University, Department of Economics. Email: ghall@brandeis.edu

[‡]Princeton University, Department of Economics. Email: jepayne@princeton.edu

[§]New York University, Department of Economics and Hoover Institution, Stanford University. Email: thomas.sargent@nyu.edu

[¶]Federal Reserve Board. Email: balint.szoke@frb.gov

1 Introduction

Determinants of interest paid on US government bonds have forever preoccupied US politicians and economists. In his 1790 *Report on Public Credit*, 34 year old Alexander Hamilton argued that the US could *lower* interest costs by restructuring political institutions to sustain tax, spending, and debt-servicing policies that would promote *expectations* among foreign and domestic lenders that the US federal government would service its debts as promised. Hamilton and other framers of the US Constitution designed arrangements to sustain subsequent federal government actions that, by reducing the substantial *default-risk* premia that in the 1780s markets assigned to debts that had been incurred by US Continental Congresses and US states to pay for the War of Independence, would allow the federal government to borrow at low interest rates in the future. To some Federal officials after Hamilton, "reducing interest payments" meant something different. Thus, when President Andrew Johnson and much of the Democratic party proposed to reduce debt servicing costs after the US Civil War, they intended to accomplish that by redefining the unit of account from gold to inconvertible paper dollars (greenbacks) that were then trading at a substantial discount relative to gold. That units-of-account sleight of hand was contested in the 1868 election. During the 1868 Presidential election campaign, the Republican party and its candidate General Ulysses S. Grant promised to sustain the practice of servicing Federal debts in gold dollars that Alexander Hamilton had proposed in 1790. Grant won. President Grant said this at his inauguration:

> A great debt has been contracted in securing to us and our posterity the Union. The payment of this, principal and interest, as well as the return to a specie basis as soon as it can be accomplished without material detriment to the debtor class or to the country at large, must be provided for. To protect the national honor, every dollar of Government indebtedness should be paid in gold, unless otherwise expressly stipulated in the contract. Let it be understood that no repudiator of one farthing of our public debt will be trusted in public place, and it will go far toward strengthening a credit which ought to be the best in the world, and will ultimately enable us to replace the debt with bonds bearing less interest than we now pay. U. S. Grant, March 1869

This paper uses a long but thin historical data set and a parameterized statistical model to infer term structures of *yields* on US Federal bonds from data that Hall et al. (2018) assembled on prices and quantities of US Federal bonds that promised to pay sequences of US dollars from 1791 to 1947. Our analysis is complicated by the presence of two types of US dollars during and after the Civil War, gold coins and greenback dollars, and a variety of monetary policy regimes.[1] Consequently, we estimate a *gold denominated* yield curve for the gold-standard period from 1791-1933 and a *greenback denominated* yield curve for the temporary and permanent deviations from the gold standard during 1862-1878 and 1933-1947 respectively. This analysis requires us to estimate gold-greenback exchange rate expectations throughout the temporary gold-standard deviation in 1862-1878. We then combine our nominal yield curve estimates with existing post-WW2 estimates and inflation data to estimate a *real* yield curve from 1791-2020.

In parametrizing and estimating a stochastic process for yield curves on US Federal bonds, we confront several challenges: (1) nineteenth century US Federal bonds often gave lenders and borrowers discretion over maturity dates, conversions, and other features, (2) nineteenth century macroeconomic data are unreliable, (3) nineteenth century US Federal bonds carried haircut risks, (4) data are sparse, (5) different bonds were

[1] George Washington and Alexander Hamilton introduced a gold standard in 1791 that was theoretically maintained until 1933, at which point Franklin Roosevelt accepted Irving Fisher's advice to abandon the gold standard. Edwards (2018) describes how the US defaulted on its promises to pay gold dollars, and Rothbard (2002) describes how Irving Fisher influenced President Roosevelt's decision to leave the gold standard. The US also temporarily departed from the gold standard from 1862-1878, during which time gold and greenback dollars exchanged at a market-determined exchange rate.

denominated in different types of dollars between 1862 and 1878 when gold and greenback dollars coexisted, and (6) we fear pricing errors. We address challenge (1) by assuming agents priced bonds under perfect foresight about discretionary contract features. Complications (2) and (3) of working with nineteenth century data guide our identification assumptions. We adopt an approach that packages default risk with a stochastic discount factor. That allows us to estimate the US Federal government's cost of bond financing using only historical bond prices and money exchange rates. A cost that we pay for this taking shortcut is that we do not directly estimate a stochastic discount factor process and don't explicitly impose absence of arbitrage opportunities.

Bond data limitations (4) and (5) guide our yield curve parametrizations. Economists at the Federal Reserve Board and other research institutions use a similar parameterization, but in inferring a yield curve from observed prices and quantities they face a different problem than we do. Because they have a superabundance of *cross-section* data on prices and quantities at each date, they solve an *overdetermined* inference problem. Our data set is too sparse along the cross-section dimension to allow us to use even a *just-identified* version of the Federal Reserve Board's procedure. To confront this data deficiency, we enlist a *prejudice* or *induction bias* in the form of a parameterized statistical model of a *panel* having scattered missing observations. Relative scarcities of data for gold denominated and greenback denominated bonds shape our approach. We have 255 gold denominated bonds in our sample (throughout the period from 1790-1933), so we parametrize the gold yield curve by modifying a time-varying version of a statistical model proposed by Nelson and Siegel (1987). This brings 7129 parameters. We have only 9 greenback denominated bonds so, instead of directly parametrizing the greenback yield curve, we construct a multiplier that transforms a gold denominated yield curve into a greenback denominated yield curve and parametrize this multiplier with a state space model of money prices. This brings 'only' 430 additional parameters. We use our parametrized statistical model to compute probabilities of parameters conditioned on our data – our way of using the data to learn about parameters that tie down posterior probability distributions of yield curves at all dates in our panel. Our data and statistical model tell us how much *smoothing* across time to do.

We approximate posterior probabilities by deploying Hamiltonian Monte Carlo and No U-Turn sampling.[2] Our data set presents technical difficulties that prevent us from applying the "standard" `Stan` toolkit: the number of observed assets change over time, the bonds have payoff streams of varying lengths, there are periods without price observations, the relevant set of bond-specific pricing errors changes over time in a complicated fashion, and we need to estimate exchange rate expectations for the sub-period 1862-1878. To tackle these difficulties, we code the log posterior function of our model from scratch in `Julia` (Bezanson et al., 2017) and feed it into the DynamicHMC.jl package by Papp et al. (2021), which is a robust implementation of the HMC-NUTS sampler that mimicks many aspects of `Stan`. An advantage of this package is that it allows the user to provide the Jacobian of the log-posterior manually and so that we don't have to rely on automatic differentiation for a model with $7,500+$ parameters. Our application of the DynamicHMC.jl package can be used for other economic models with tractable likelihood functions that don't easily fit into the `Stan` framework.

Our responses to challenges (1)-(5) assemble a collection of assumptions that allow us to derive and estimate our asset pricing equations. A final challenge (6) arises because there are good reasons to believe that particular bonds may violate these assumptions in some periods. We address this issue by introducing *bond-specific* pricing errors. This approach decreases influences of peculiar bonds on our yield estimates while still informing us about situations when our collective assumptions prevent us from consistently pricing our *cross-section* of bonds using our pricing formulas. We use these errors to refine our characterization and selection of the US Federal bonds used to estimate yield curves. We view this error inspection approach as being in the spirit of Hansen and

[2]Hamiltonian Monte Carlo is named after mathematician and physicist William R. Hamilton, not US Secretary of the Treasury Alexander Hamilton.

Jagannathan (1997).

We can use our estimated US Federal debt yield curve to supplement informal historical stories about times of fiscal stress by adding numbers and measures of our uncertainty about them. By listening to how our data speak to us through our economic-statistical model, we detect patterns in 19th century yield curves that set the stage for comparisons with 20th and 21st century patterns. Some of our findings are:

- Our estimates of yields differ substantially from widely used ones, especially during some episodes of fiscal stress.
- While yield curves usually sloped upward, "inversions" occurred during major wars, the late 1820s, the mid 1890s, and before the Great Contraction that began in 1929.
- Yields have trended downward since 1790.
- US yields initially exceeded yields on British consols, but by 1880 the spread had vanished. It seems to have taken most of the nineteenth century for US debt to become "safe".
- During the Civil War suspension of gold convertibility, creditors initially expected a rapid return to par. They became less certain about that in the post-Civil War period. What seemed to be a strong "nominal anchor" during the Civil War apparently weakened in the post-war period after the US did not immediately return to the gold standard.
- Our yield curves allow us to approximate the market value of marketable federal debt and improve widely used estimates of debt-to-GDP ratios in the 19th century.
- Our nineteenth century real yield curves belie an inference drawn in widely cited regression studies of historical US time series that surges of government expenditures and deficits associated with big wars were not accompanied by higher yields on US government debt.
- The US fought more wars after costs of financing war time surges in government purchases dropped.

Related Work

Homer and Sylla (2004) construct time series of yields to maturity on 10-year Federal treasuries, New England Municipal Bonds, and Corporate Bonds.[3] The closest counterpart to ours is their yield to maturity on 10-year treasuries, which they compute as coupon rated on US federal bonds that have approximately 10 years to maturity and trade close to par. Estimating yield curves allows us to compute term spreads. It also allows us to fill in the Homer and Sylla (2004) 10-year yield series during periods like the Civil War, when they are unable to apply their methodology. Remarkably, Homer and Sylla (2004)'s 10-year Federal treasury series is often not the long-term US bond series used by economic historians. Instead, researchers (e.g. Officer and Williamson (2021), Shiller (2015), Jordà et al. (2019), and Hamilton et al. (2016)) have typically used a 'composite series' that combines the Homer and Sylla (2004) estimates for the period from 1798-1861 with the yield-to-maturity on a set of the New England municipal bonds for the period 1862-1899 and the yield-to-maturity on corporate bonds for the period 1900-1940. We provide evidence that this blended series underestimates costs of government financing during the Civil War and overestimates those costs after the War because it calculates yields on high-grade state or corporate bonds rather than on US Federal debt.[4] Technically, our work is related to Svensson (1995), Dahlquist and Svensson (1996), Cecchetti (1988), Annaert et al. (2013), Andreasen et al. (2019), Diebold

[3]A yield to maturity is also called an internal rate of return.
[4]See Siegel (1992), in particular Section 2.2.

and Li (2006) and Diebold et al. (2008) who like Gürkaynak et al. (2007) and ourselves implement versions of the parametric yield curve model of Nelson and Siegel (1987). Our analysis of events during the greenback period from 1862 to 1879 revisits issues in landmark studies of Mitchell (1903, 1908) and Roll (1972).

Section 2 describes data. Section 3 provides a glossary for notation. Section 4 outlines a theory of zero-coupon bond yields and describes how we parametrize yield curves. Section 5 delineates our econometric strategy. Section 6 discusses statistical inferences about gold denominated yield curves. Section 7 discusses statistical inferences about greenback denominated yield curves and gold-greenback price expectations during and after the Civil War. Section 8 discusses statistical inferences about real yield curves. Section 9 concludes.

2 Data and Context

We describe our data, provide historical context on characteristics of 19th century US federal monies and bonds, and outline challenges that these characteristics pose for yield curve estimation. These challenges shape specification and estimation strategies deployed in Sections 4 and 5.

2.1 Data Sources

We have assembled prices, quantities, and descriptions of all securities issued by the US Treasury between 1776 and 1960. Figures 14 and 15 (at the end of the paper) summarise issue sizes, durations, and coupon rates. The full data set is available at the Github repository `https://github.com/jepayne/US-Federal-Debt-Public` and construction methods are explained in Hall et al. (2018).[5] In this section, we spotlight decisions about our data that we made to prepare for the econometrics presented in this paper.

Our bond price data are at a monthly frequency. When available, we use the closing price at the end of each month. However, if a closing price is not available, then we use an average, bid, or ask price (in that order of precedence)[6]. The primary sources for the price data from 1776 to 1839 are Razaghian (2002) and Sylla et al. (2006). Prices from 1840 to 1899 are from Razaghian (2002), Martin (1886), the *Merchants' Magazine and Commercial Review*, the *Commercial and Financial Chronicle*, the *New York Times*, and *Global Financial Data*. Prices from 1900 to 1918 are from the *Commercial and Financial Chronicle* and US Treasury Circulars. When overlaps occurred, data were taken from the US Treasury Circulars. Prices from 1919 to 1925 are from "United States Govt. Bonds" tables in the New York *Times*. Prices after 1925 are taken from the *CRSP US Treasury Database*.[7]

The quantity data are quarterly from 1776 to 1871 and monthly thereafter. All quantity entries record the quantity outstanding on the last business day of the period. The quantities outstanding from 1790 to 1871 are imputed from the issue and redemption series reported by Bayley (1882). We cross-checked these quantities against quantity outstanding series reported in Register's Office (1886). After 1871 our source for quantity outstanding series is the United States Department of the Treasury (2015) *Monthly Statements of the Public Debt*. The call data are from *Annual Reports of the Secretary of Treasury* for various years. Data on Treasury securities held in government accounts are from *Banking and Monetary Statistics 1914-1941* prior to 1941 and from *Treasury Bulletin* thereafter.[8]

Data on contractually promised dollar payments come from Bayley (1882) for the period from 1790-1871 and from United States Department of the Treasury (2015) *Monthly Statements of the Public Debt* for the period from 1872-1960.

[5] Only data from publicly available data sets are posted on the GitHub page.
[6] The order of precedence is chosen based on data availability
[7] See `http://www.crsp.com/products/research-products/crsp-us-treasury-database`.
[8] See Board of Governors of the Federal Reserve System (1943) and Register's Office (1886).

We require data on greenback-gold dollar exchange rates to estimate the greenback and real yield curves. For the gold to greenback exchange rate, we use Greenback price data from Mitchell (1908)[9] for the period from 1862-1878 during which both greenback and gold dollars circulated. For the gold to goods exchange rate, we combine a number of series. For the period from 1800-1860, we use the wholesale price index from Warren et al. (1932). For the period from 1860-1913, we use the General Price Level Index from the *NBER Macroeconomics Database*[10]. For the period from 1913-2020, we use the Consumer Price Index from the U.S. Bureau of Labor Statistics.

2.2 19th Century Dollars

Between April 1792 and February 1862, the US dollar was defined in terms of gold and silver.[11] Except for the activities of the monopoly First (1791-1811) and Second (1816-1836) Banks of the United States, the federal government issued no paper notes, only gold and silver coins. Private state-chartered banks issued paper notes convertible into gold on demand. In January 1862, these banks stopped honoring their legal obligation to convert their notes into specie (they "suspended" convertibility).

On February 25, 1862, Congress passed a Legal Tender Act that authorized the Treasury to issue 150 million dollars of a paper currency known as greenbacks that the government did not promise immediately to exchange for gold dollars. Subsequent acts authorized the Treasury to issue more notes, eventually totalling 450 million dollars. Investors could use greenbacks to purchase bonds from the federal government at their par values. Gold dollars continued to be used for settling international transactions and for paying US tariffs. From 1862 to December 31, 1878 paper notes ("greenbacks" or "lawful money") traded at a discount relative to gold dollars ("gold" or "coin"). Figure 1 plots the greenback to gold exchange rate[12] and the prices of outstanding bonds. The greenback depreciated in value substantially during the Civil War and did not attain parity with gold until January 1, 1879 when the US Treasury stood ready to convert dollars into gold dollars one-for-one. Convertibility between gold and paper currency at par prevailed until 1933 when Franklin Roosevelt increased the paper price of gold and prohibited private US citizens from holding gold coins.

2.3 19th Century US Federal Bonds

Before World War I, the federal government issued bonds infrequently and quantities of new bonds issued were often small. The US Congress, rather than the Treasury, designed each government security with the consequence that securities varied over time in terms of their coupon rates, denominations, lengths, units of account, tax exemptions, and call features. Before the 1920s, the federal government occasionally issued customized long term debt, mostly to finance specific infrastructure projects, debt reschedulings, and wars. As a result, between 1776 and World War I, the US Congress only authorized the Treasury to issue a total of approximately 200 distinct securities, with at most 8 distinct ones being authorized in any one year.

From 1917 to 1939, Congress progressively delegated all decisions about designing US debt instruments to the Treasury and the Treasury gradually standardized security design. As a result, from 1920 to 1960 alone, the Treasury issued about 2500 securities with a wide range of maturities. Ultimately, this transformed the market for US Treasury securities into the world's most liquid debt market with a collection of standardized securities at many maturities that allowed a large national debt to be issued and rolled over, seemingly perpetually.

[9] See Table 2

[10] See `https://www.nber.org/research/data/nber-macrohistory-iv-prices`

[11] Prior to 1792, a dollar referred to a Spanish silver coin.

[12] The exchange rate is stated as the number of greenback dollars required to purchase 100 gold dollars.

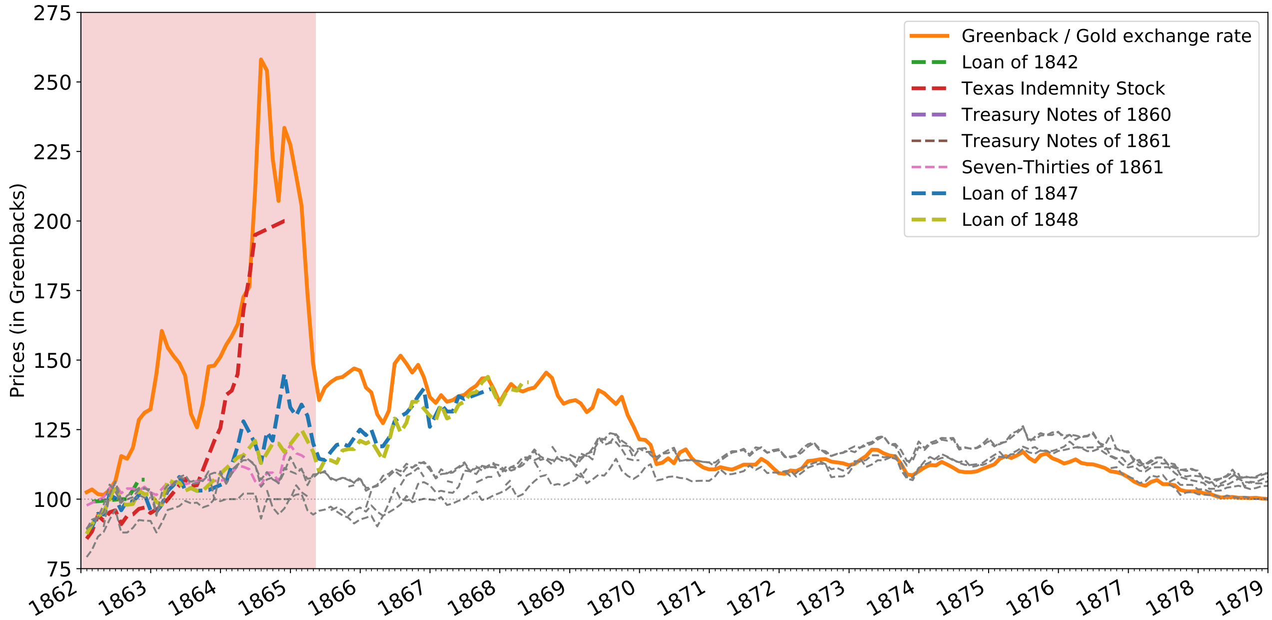

Figure 1: Prices of Gold and Bonds: 1860-1880.

The solid orange line depicts the greenback to gold exchange rate (expressed as the number of greenback dollars required to purchase 100 gold dollars). The dashed lines depict observed prices (denominated in greenbacks) for the outstanding bonds. The light red interval depicts the Civil War.

2.4 Estimation Challenges

Skilled researchers have estimated yield curves on US federal debt in the post-WW2 period, by which time Federal debts had became standardised and government bonds for sale had become plentiful. We estimate yield curves starting in 1791 and so have to confront estimation challenges posed by peculiar structures of US Federal bond markets before 1920. This requires us to address the following issues.

(1) *How should we handle peculiar bonds?*

Throughout our sample, many US Treasury securities had custom features such as flexible maturities and conversion options. In principle, we could attempt to construct custom pricing formulas for each bond using the universe of possible derivatives. In practice, we start by arbitrarily "converting" idiosyncratic contract features into those of more standard bonds. We do this by imposing perfect foresight about the discretionary components of the contracts. For bonds that specified a window during which the government could call the bond or bond holders could the redeem the bond, we impose that agents were pricing the bonds knowing the future redemption date.[13] For bonds that could be converted into other bonds, we impose that agents were pricing bonds knowing the cash flows of the new bond after conversion. We refine the perfect foresight assumption by studying bond pricing errors.

(2) *How should we handle periods that provide sparse or inaccurate macroeconomic data?*

[13] More specifically, we impose perfect foresight in the following way. For callable bonds, we set the maturity date to the date at which the government called the bond. For redeemable long term bonds (greater than 3 years maturity) and/or bonds that pay regular coupons, we set the maturity date to the last date at which any of the bonds could be redeemed. For redeemable short term bonds (less than 3 years maturity) that pay coupons at maturity, we set the maturity date to the last date at which bonds were issued plus the duration of the bond and match total coupon payments with bond duration. For example, for a 1-year bond, this means we impose that only one year's coupons are paid at redemption, regardless of the date at which the bond was redeemed.

In principle, we could attempt to use historical macroeconomic data to estimate a model of the stochastic discount factor that prices macroeconomic risks. However, we are skeptical about the quality of nineteenth century macroeconomic data, especially at high frequency. For this reason, we start by estimating a "theory-lite" model that sidesteps directly specifying a stochastic discount factor process. As we outline in section 4, this approach still allows us to estimate the US Federal government's cost of finance via debt issues without recovering a stochastic discount factor. We leave for a future work an attempt to use noisy historical macroeconomic data to estimate directly a stochastic discount factor process.

(3) *How should we handle haircut risk?*

Today, US federal debt is often assumed to carry no haircut risk. That assumption is implausible for much of the nineteenth century. Although the US federal government never officially imposed haircuts on debt holders, it faced several crises during the nineteenth century (e.g., the War of 1812 and the Civil War) that threatened to leave the US federal government unable or unwilling to repay its debts. State governments also recurrently defaulted during the nineteenth century, and the Confederate States of America defaulted. We address this difficulty by packaging haircut risk with a stochastic discount factor, by imposing that haircut risk is common across government bonds, and ultimately by estimating prices of risky government promises. We tell how we do this in section 4.1.

(4) *How should we handle periods with sparse bond data?*

The Federal government issued few securities during the 19th century so that we have a limited number of price observations. We can see this in Figure 2, which shows monthly time series for the number of securities with observed prices and times to maturity (in years) of all outstanding bonds. The gold color scheme represents gold denominated bonds. The green color scheme represents greenback denominated bonds. The gray color scheme represents the five-twenties, which had an ambiguous denomination. There were often fewer than 5 price observations in a given period, often only for bonds with long maturities. We have no prices in the late 1830s because there were no federal securities outstanding then. This means that while we have "big data", our unbalanced sample prevents us from applying commonly used techniques from the yield curve estimation literature (e.g., Gürkaynak et al. (2007) take advantage of the abundance of post-WW2 data to estimate a yield curve *day-by-day*). Instead, we must posit a statistical model that lets us learn about yields at all dates simultaneously by pooling information across time periods. We do this by first imposing a time-varying Nelson and Siegel (1987) style parametrization of the gold denominated yield curve in section 4.2 and then "smoothing" parameter updates in section 5.1.

(5) *How should we handle greenback denominated bonds, 1862-1872?*

When gold and greenback dollars coexisted (1862-1872), different US Treasury bonds promised payments in different currencies. Some bonds promised all payments in gold (we refer to these as "gold" denominated bonds); other bonds promised all payments in greenbacks (we refer to these as "greenback" denominated bonds); and yet other bonds offered coupons in gold but left ambiguous whether the principal would be paid in gold or greenbacks (we refer to these as "ambiguously" denominated bonds). While bonds denominated in different currencies present an opportunity because they allow us to estimate both gold and greenback denominated nominal yield curves, they create additional challenges. One difficulty is that we observe only 9 greenback denominated bonds and only 6 ambiguously denominated bonds. This means that our sparse cross-section problem is worse for estimating the greenback yield curve. Consequently, we use information from the gold denominated bonds to help estimate the greenback yield curve. We do this in section 4.3, where we construct a multiplier process that transforms the gold denominated yield curve into a greenback denominated yield curve; we describe the evolution of the multiplier with a parameterized state-space model of dollar prices and parametrize the relationship between the multiplier and the gold yield curve by imposing that inflation is independent of haircut risk and real variables.

(6) *How should we treat pricing errors?*

Our responses to the previous questions impose a collection of assumptions: perfect foresight about discretionary contract components, common haircut risk and/or convenience yields across all government bonds, and independence between inflation and haircut risks. These assumptions allow us to derive and estimate our asset pricing equations. However, there are good reasons to think that in some periods particular bonds violate these assumptions. We address this issue in section 5.2 by introducing *bond-specific* price errors. This decreases the influence of peculiar bonds on our yield estimates while still informing us about situations when our collective assumptions prevent us from consistently pricing our *cross-section* of bonds using our pricing formulas. We view this error inspection approach as being in the spirit of Hansen and Jagannathan (1997).

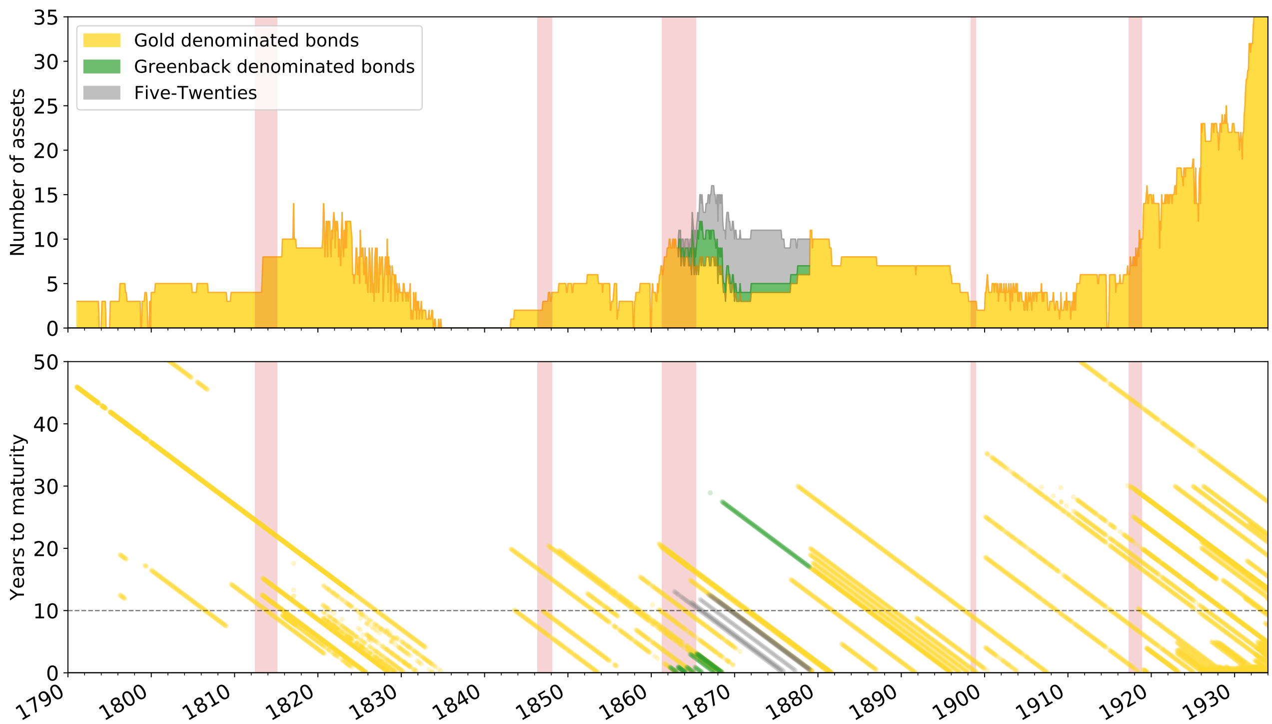

Figure 2: The top panel depicts the number of securities with observed prices each month. The bottom panel depicts maturities (in years) of observed securities. Darker lines indicate overlapping securities. Red bars correspond to wars.

3 Glossary

i Index for a particular bond. 10

j Number of periods until a bond payout. 10

n Index for type of dollar: $n = g$ indicates gold dollars, $n = d$ indicates greenback dollars, $n = a$ indicates denomination is ambiguous. 11

$\overline{m}_{t+j}^{(i,n)}$ Number of dollars of type n promised by bond i at time $t + j$. 11

$m_{t+j}^{(i,n)}$ Number of dollars of type n actually paid by bond i at time $t + j$. 11

$\xi_{t+j}^{(i,n)} := m_{t+j}^{(i,n)} / \overline{m}_{t+j}^{(i,n)}$ Fraction of promised payment of dollars of type n that is actually paid. 11

$c_{t+j}^{(i,n)}$ Amount of composite consumption good delivered by bond i at time $t + j$. 11

$p_t^{(i,j,n)}$ Price (in units of gold) of a zero-coupon bond i that promises $\overline{m}_{t+j}^{(i,n)}$ dollars of type n at time $t + j$. 11

$q_t^{(j,n)}$ Price (in dollars of type n) of a zero-coupon bond i that promises 1 dollar of type n at time $t + j$. 12

$z_t^{(j)}$ Multiplier that converts gold price $q_t^{(j,g)}$ into greenback price $q_t^{(j,d)}$. 14

$w_t^{(j)}$ Multiplier that converts gold price $q_t^{(j,g)}$ into ambiguous price $q_t^{(j,a)}$. 43

S Stochastic discount factor (SDF) process. 11

$e_t^{(n)}$ Amount of composite consumption good required to purchase 1 unit of dollar of type n (i.e., goods price of dollar of type n). 11

$P_t := e_t^{(d)} / e_t^{(g)}$ Number of gold dollars paid for 1 greenback dollar at time t (i.e., the gold price of greenbacks at time t). 11

4 Parameterized Yield Curves

At a given date, a term structure of interest rates is a list of yields on zero-coupon bonds of maturities $j = 1, 2, \ldots, J$. We approximate term structures of gold denominated, greenback denominated, and real (i.e., goods-denominated) US Federal securities. Because the US government did not issue zero-coupon securities of all maturities, we must approximate zero-coupon yield curves on US Treasury debt indirectly from observed prices of a limited set of federal government securities having differing coupons, par values, and maturity dates. As discussed in section 2, inference is especially challenging before World War I because the Treasury issued bonds infrequently; bonds contained default risk; and bonds differed on whether they promised to pay gold dollars, silver dollars, or greenbacks that floated relative to gold and silver dollars, their tax exemptions, their investor redemption options, and government call options. To make progress, we impose a collection of assumptions that allow us to parametrize stochastic processes for yield curves on US Federal bonds. In section 4.1, we outline how we package the stochastic discount factor with haircut risk rather than estimating a stochastic discount factor process directly. In sections 4.2 and 4.3, we outline our parametrizations of gold denominated and greenback denominated yield curves. Data limitations shape these parametrizations. In section 4.4, we interpret identifying assumptions in terms of risk premia. In section 4.5, we bring all of our assumptions together to construct pricing equations for the yield curve.

4.1 Zero-Coupon Federal Government Bonds

Consider a setting with a composite consumption good, gold dollars, greenback dollars and a collection of zero-coupon bonds, some of which promise payouts in gold dollars, other of which promise greenback dollars. We use gold dollars as numeraire. Suppose that a particular j-maturity zero-coupon bond, indexed by i, promises to pay $\overline{m}_{t+j}^{(i,n)}$ units of type n dollars at time $t+j$, where $n \in \{g, d\}$ denotes gold (g) or greenback (d) dollars. To acknowledge default risk and currency risk, we denote a bond's *ex post* actual payment of n dollars in period $t+j$ by $m_{t+j}^{(i,n)}$ and the bond's actual payment in terms of consumption goods as $c_{t+j}^{(i,n)}$. Let $p_t^{(i,j,n)}$ denote the price of such a bond in units of gold dollars. Let $e_t^{(n)}$ denote the quantity of consumption goods that can be exchanged for one dollar of type n at time t (i.e., the consumption goods price of dollars of type n). Let $P_t := e_t^{(d)}/e_t^{(g)}$ denote the quantity of gold dollars that can be exchanged for 1 greenback dollar at time t (i.e., the price of greenbacks in terms of gold dollars at time t). We invoke two assumptions throughout.

Assumption 1. For each $t \geq 0$, there exists a non-negative stochastic discount factor (SDF) stochastic process $S^{(t)}$ that can price all government bonds.

This peculiar technical assumption lets us adapt some formulas from asset pricing theory. We specify a separate SDF process at each t rather a single SDF to ensure that our asset pricing formulas are compatible with the version of the Nelson and Siegel (1987) parametrization that we adopt in section 4.2.[14] With some abuse of notation, we drop the explicit superscript on the SDF to simplify the expressions. The stochastic process at time t verifies:

$$e_t^{(g)} p_t^{(i,j,n)} = \mathbb{E}_t \left[\left(\frac{S_{t+j}}{S_t} \right) c_{t+j}^{(i,n)} \right]$$

where $e_t^{(g)} p_t^{(i,j,n)}$ is the goods price of the zero-coupon bond, the SDF S_t is measured in time 0 consumption goods per time t, and $\mathbb{E}_t$ denotes a mathematical expectation conditional on time t information. A bond's payout of consumption goods satisfies $c_{t+j}^{(i,n)} = e_{t+j}^{(n)} m_{t+j}^{(i,n)}$, so that we can rewrite the preceding equation in terms of claims on dollars of type $n = g$ and $n = d$:

$$p_t^{(i,j,n)} = \begin{cases} \mathbb{E}_t \left[\left(\frac{S_{t+j}}{S_t} \right) \left(\frac{e_{t+j}^{(g)}}{e_t^{(g)}} \right) \xi_{t+j}^{(i,g)} \right] \overline{m}_{t+j}^{(i,g)}, & \text{if } n = g \\ P_t \mathbb{E}_t \left[\left(\frac{S_{t+j}}{S_t} \right) \left(\frac{e_{t+j}^{(d)}}{e_t^{(d)}} \right) \xi_{t+j}^{(i,d)} \right] \overline{m}_{t+j}^{(i,d)}, & \text{if } n = d \end{cases}$$

where we define a haircut fraction $\xi_{t+j}^{(i,n)} := m_{t+j}^{(i,n)} / \overline{m}_{t+j}^{(i,n)} \in [0,1]$ and we are entitled to move $e_t^{(n)}$ and $\overline{m}_{t+j}^{(i,n)}$ outside the conditional expectation operator because they are known conditional on time t information.

For convenience, we define:

$$q_t^{(i,j,n)} := \mathbb{E}_t \left[\left(\frac{S_{t+j}}{S_t} \right) \underbrace{\left(\frac{e_{t+j}^{(n)}}{e_t^{(n)}} \right)}_{\substack{\text{Currency } n \\ \text{inflation risk}}} \underbrace{\xi_{t+j}^{(i,n)}}_{\substack{\text{Haircut} \\ \text{risk}}} \right], \tag{4.1}$$

which is the price of a promise to one dollar of type n at time $t+j$ packaged with bond i's haircut risk; it is expressed in type n dollars. Observe that the right side of equation (4.1) prices two risks borne by bond holders: risk in the goods value of currency n (captured by $e_{t+j}^{(n)}/e_t^{(n)}$) and haircut risk (captured by $\xi_{t+j}^{(i,n)}$). We can then

[14] Several authors have studied how to reconcile time varying versions of the Nelson and Siegel (1987) parametric yield curve specification with an arbitrage free asset pricing theory and a single SDF process that would be associated with it. Our model is closest to Diebold et al. (2005), which imposes an autoregressive structure on the NS factors. Krippner (2015) shows that this specification is an "eigenvalue approximation" to an arbitrage free affine asset pricing model, but Björk and Christensen (1999) and Filipović (1999) argue that this approximation cannot be formalized. To deal with this issue, we assume a sequence of stochastic discount factor processes. An alternative approach would be to use an augmented yield curve specification proposed by Christensen et al. (2011). We do not choose this alternative approach here because we want to stay close to the Diebold et al. (2005) specification that is more commonly used in applied work.

express bond i's price as:

$$p_t^{(i,j,n)} = \begin{cases} q_t^{(i,j,g)} \overline{m}_{t+j}^{(i,g)}, & n = g \\ P_t q_t^{(i,j,d)} \overline{m}_{t+j}^{(i,d)}, & n = d \end{cases}$$

Notation Aside: So far, we have used real stochastic discount factor processes S . In the spirit of Piazzesi and Schneider (2007), we can define a dollar n stochastic discount factor by:

$$S_t^{(n)} := S_t e_t^{(n)}, \quad \forall t \geq 0$$

whose units are time 0 dollars of type n per time t dollar of type n. Zero-coupon bond prices satisfy:

$$q_t^{(i,j,n)} = \mathbb{E}_t \left[\underbrace{\left(\frac{S_{t+j}^{(n)}}{S_t^{(n)}} \right)}_{\substack{\text{Dollar } n \\ \text{SDF}}} \underbrace{\xi_{t+j}^{(i,n)}}_{\substack{\text{Haircut} \\ \text{risk}}} \right],$$

Assumption 2. For government securities, haircut fraction $\xi_t^{(i,n)}$ is independent of i and n at any time t so that we write $\xi_t^{(i,n)} = \xi_t$, for all government bonds i, dollar types n, and times t.

This assumption says that, within each time period, there is no cross-sectional variation in government haircut risk because the government imposes the same haircut on all bonds outstanding at a time of default. As a result, the price of a federal government promise to a dollar of type n is independent of i and is denoted $q_t^{(j,n)}$. We need this assumption to identify $q_t^{(j,n)}$ because, without it, the number of asset pricing equations would equal the number of observed government bond prices. Zero-coupon government bond prices can thus be expressed as:

$$p_t^{(i,j,n)} = \begin{cases} q_t^{(j,g)} \overline{m}_{t+j}^{(i,g)}, & n = g \\ P_t q_t^{(j,d)} \overline{m}_{t+j}^{(i,d)}, & n = d \end{cases}$$

To transform zero-coupon bond prices into yields, we let $\mathsf{y}_t^{(n)} := \{y_t^{(j,n)}\}_{j=0}^{\infty}$ denote a type n dollar yield curve in period t whose j-th component is

$$y_t^{(j,n)} := -\frac{\log q_t^{(j,n)}}{j}.$$

As discussed in section 2, we do not directly estimate the SDF process, $\{S^{(t)}\}_{t \geq 0}$, because we cannot identify the SDF using only the bond price and money exchange rate data we have available. Instead, we parametrize and estimate the yields, $\mathsf{y}_t^{(n)}$, which means that we parametrize and estimate prices $\mathsf{q}_t^{(n)}$ of risky government promises. Evidently, we price these bonds using a formula that combines an SDF with dollar n inflation risk and haircut risk.

Aside: Did Congress issue revenue bonds? Until the Second Liberty Bond Act of 1917, Congress sequentially directed the Treasury to sell particular securities, one issue at a time, and restricted spending the proceeds on specific purposes (e.g., infrastructure, refinancing of old debt, military expenditures). However, Congress has always allowed the Treasury to use general revenue to make coupon and principal payments rather than dedicate specific revenue sources (e.g., tolls or land sales). In the language of municipal finance, Treasury bonds were and continue to be general obligation bonds, not revenue bonds. Consequently, had the Treasury explicitly defaulted on its debt, it is not clear that some securities would have had more seniority than others. For this reason, we start with assumption 2 and then inspect pricing errors for violations.

4.2 Parametrization of Gold Dollar Yield Curves

Here and in the following subsection, we impose additional assumptions to facilitate approximating gold and greenback yield curves. Data limitations influence our assumptions. Because the US government issued many bonds denominated in gold dollars, we can estimate gold denominated yield curves by imposing the yield curve parameterization outlined in assumption 3. Because the US government issued few bonds denominated in greenback dollars, to estimate greenback yield curves we impose further assumptions to be described in the next subsection.

Assumption 3. The j-th component $y_t^{(j,g)}$ of a gold denominated yield curve takes the form

$$y_t^{(j,g)} = \beta_{0,t} + (\beta_{1,t} + \beta_{2,t}) \left[1 - \exp\left(-\frac{j}{\tau}\right)\right] / \left(\frac{j}{\tau}\right) - \beta_{2,t} \exp\left(-\frac{j}{\tau}\right). \tag{4.2}$$

with parameters $\beta_t := [\beta_{0,t}, \beta_{1,t}, \beta_{2,t}]'$ and τ.

This specification follows Nelson and Siegel (1987) and has a number of desirable features. First, it is flexible enough to generate "typical yield curve shapes" (e.g., monotonic, humped, and S-shaped curves). Second, it ensures that yields converge as maturity goes to $+\infty$, with $\beta_{0,t}$ parameterizing the asymptote. Third, separate parameters shape different parts of the yield curve: $\beta_{1,t}$ at the short end of the yield curve, $(\beta_{2,t}, \tau)$ for medium-term yields. For simplicity, we collect the four yield curve parameters into the vector

$$\tilde{\beta}_t := \left[\beta_{0,t},\ \beta_{1,t}, \beta_{2,t},\ \tau\right]'.$$

Fourth, it is compatible with estimates of recent yield curves.[15] Finally, it implies convenient formulas for a gold denominated forward rate curve:

$$f_t^{(j,g)} = \beta_{0,t} + \beta_{1,t} \exp\left(-\frac{j}{\tau}\right) + \beta_{2,t} \exp\left(-\frac{j}{\tau}\right)\left(\frac{j}{\tau}\right)$$

Assumptions 1, 2, and 3 are sufficient to estimate gold dollar denominated yield curves if we restrict our sample to gold denominated bonds. For this reason, we treat gold yield curves as our "baselines" throughout the paper and then impose additional assumptions to restrict how greenback denominated bonds relate to these baseline yield curves (in section 4.3).

4.3 Parametrization of Greenback Dollar Yield Curves

Assumption 3 parameterizes gold dollar yield curves but not greenback yield curves. Perhaps we could have parameterized greenback yield curves in a similar way, but because we have far fewer price observations for greenback denominated bonds than gold dollar denominated bonds, we did not. We instead proceed by adding assumptions that let us use information about gold denominated bonds to help us estimate greenback yield curves.

Assumption 4. Conditional on time t information, S_{t+j} and ξ_{t+j} are independent of $e_{t+j}^{(g)}$ and $e_{t+j}^{(d)}$.

[15] For example, Gürkaynak et al. (2007) use this form for the period 1961-1980. After 1980, they use an extension proposed by Svensson (1994) to allow for a second hump in the yield curve regarded as a "convexity effect".

In the next subsection, we interpret this assumption in terms of risk premia . Here, we show how this assumption links greenback dollar denominated yield curves to gold dollar denominated yield curves. Evidently:

$$
\begin{aligned}
q_t^{(j,d)} &= \mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\frac{e_{t+j}^{(g)}}{e_t^{(g)}}\right)\xi_{t+j}\left(\frac{P_{t+j}}{P_t}\right)\right] \\
&= \frac{1}{e_t^{(g)}P_t}\mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\xi_{t+j}\right]\mathbb{E}_t\left[e_{t+j}^{(g)}P_{t+j}\right] \\
&= \frac{q_t^{(j,g)}}{P_t}\left(\mathbb{E}_t[P_{t+j}] + \frac{\mathrm{Cov}_t\left(e_{t+j}^{(g)}, P_{t+j}\right)}{\mathbb{E}_t\left[e_{t+j}^{(g)}\right]}\right)
\end{aligned}
$$

where in the second line we used the simplifications afforded by assumption 4 to decompose the conditional expectation. We define a conversion multiplier $z_t^{(j)}$ by

$$
z_t^{(j)} := \mathbb{E}_t[P_{t+j}] + \frac{Cov\left(e_{t+j}^{(g)}, P_{t+j}\right)}{\mathbb{E}_t\left[e_{t+j}^{(g)}\right]} \tag{4.3}
$$

that verifies

$$
q_t^{(j,d)} = q_t^{(j,g)}\frac{z_t^{(j)}}{P_t}.
$$

We can express prices of zero coupon bonds as:

$$
p_t^{(i,j,n)} = \begin{cases} q_t^{(j,g)}\overline{m}_{t+j}^{(i,g)}, & \text{if } n = g \\ P_t q_t^{(j,d)}\overline{m}_{t+j}^{(i,d)} = q_t^{(j,g)} z_t^{(j)}\overline{m}_{t+j}^{(i,d)}, & \text{if } n = d \end{cases}
$$

Observe that price $p_t^{(i,j,n)}$ is in units of gold dollars at time t, price $q_t^{(j,g)}$ is in units of t-period gold dollars per time $(t+j)$-period gold dollar, conversion multiplier $z_t^{(j)}$ is in units of $(t+j)$-period gold dollars per unit of $(t+j)$-period greenback dollars, and $\overline{m}_t^{(i,d)}$ is in units of $(t+j)$-period greenback dollars. Thus, the adjustment factor $z_t^{(j)}$ converts greenback dollars to gold dollars in a way that acknowledges greenback-gold dollar exchange rate risk.

To work with $z_t^{(j)}$, we must know the period-t conditional expectation of future paths of P_{t+j}, the period-t conditional expectation of future paths of $e_{t+j}^{(g)}$, and a period-t conditional covariance between P_{t+j} and $e_{t+j}^{(g)}$. We parametrize these conditional moments by adopting a simple yet flexible statistical model for exchange rates.

Assumption 5. Joint dynamics of exchange rates $v_t := [P_t, e_t^{(g)}]'$ obey a state-space model:

$$
\begin{aligned}
v_{t+1} &= \mu_t + x_t + F\varepsilon_{v,t+1} \\
x_{t+1} &= A_t x_t + K\varepsilon_{v,t+1}
\end{aligned}
\qquad \varepsilon_{v,t+1} \sim \mathcal{N}\left(\mathbf{0}, \mathbb{I}_2\right), \ \forall t \geq 0 \tag{4.4}
$$

where x_t is a 2-vector hidden state with given initial x_0, F and K are 2×2 matrices with F being lower triangular. Parameters μ_t and A_t are allowed to be time varying. We collect the parameters from model (4.4) into the vector:

$$
\zeta_t := [\mu_t', \ \mathrm{vec}(A_t)', \ \mathrm{vec}(F)', \ \mathrm{vec}(K)']'.
$$

Using the statistical model to compute the conditional moments that constitute the adjustment factor $z_t^{(j)}$, we can write $z_t^{(j)} = z^{(j)}(\zeta_t)$.

Discussion of time varying μ_t and A_t. Following Cogley and Sargent (2005) and Cogley (2005), we interpret time-variation in ζ_t as bondholders' "changing beliefs" about future values of v_t induced by shifts in fiscal-monetary policy rules. During and after the Civil War, the direction of US monetary-fiscal policies recurrently either shifted markedly or seemed to be on the verge of swerving onto another course. Dewey (1922, pp. 340–352) described unfoldings of political struggles about how and whether to service or to tax bond holders or outright to default on US bonds. After describing tentative steps initially taken in early 1866 to retire greenbacks, on page 340 Dewey wrote that "... a great opportunity was lost, for public sentiment in the winter of 1866 would have sustained a more rapid contraction; the country at large was expecting it, and the deed might have been accomplished if Congress had had enough courage." We cope with this situation by positing a shifting law of motion for the relative value of greenback dollars. We assume that financial market participants understood that policies were drifting and sought to adapt their beliefs accordingly. The vector ζ_t represents their period-t beliefs about the currency price processes; ζ_t drifts as they learn. We assume that the pricing formulas in (4.5) hold on a date-by-date basis, i.e., although agents keep updating their beliefs, they treat the updated ζ_t as if it would remain constant forever. Kreps (1998) incorporates such behavior in his 'anticipated utility' model.

4.4 Currency Risk Premia

To elaborate assumption 4, we express yields in terms of currency risk premia. Define a risk-free real price and risk-free real yield by:

$$\hat{q}_t^{(j)} := \mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right], \qquad \hat{y}_t^{(j)} := -\frac{\log \hat{q}_t^{(j)}}{j}$$

Lemma 1. *The difference between a dollar n yield and the risk free real yield is approximately:*

$$\begin{aligned}
y_t^{(j,n)} - \hat{y}_t^{(j)} \approx\ & \underbrace{-\frac{1}{j}\log\left(\frac{\mathbb{E}_t\left[e_{t+j}^{(n)}\right]}{e_t^{(n)}}\right)}_{\text{Expected dollar } n \text{ inflation}} + \underbrace{\frac{-1}{j}\log \mathbb{E}_t\left[\xi_{t+j}\right]}_{\text{haircut probability}} \\
& + \underbrace{-\frac{1}{j}\left(Cov_t\left(\frac{\xi_{t+j}}{\mathbb{E}_t\left[\xi_{t+j}\right]}, \frac{e_{t+j}^{(n)}/e_t^{(n)}}{\mathbb{E}_t[e_{t+j}^{(n)}/e_t^{(n)}]}\right)\right)}_{\text{Risk from haircut \& inflation comovement}} + \underbrace{-\frac{1}{j}\left(Cov_t\left(\frac{S_{t+j}/S_t}{\mathbb{E}_t\left[S_{t+j}/S_t\right]}, \frac{\xi_{t+j}}{\mathbb{E}_t\left[\xi_{t+j}\right]}\right)\right)}_{\text{Risk premium on haircut risk}} \\
& + \underbrace{-\frac{1}{j}\left(Cov_t\left(\frac{S_{t+j}/S_t}{\mathbb{E}_t\left[S_{t+j}/S_t\right]}, \frac{e_{t+j}^{(n)}/e_t^{(n)}}{\mathbb{E}_t\left[e_{t+j}^{(n)}/e_t^{(n)}\right]}\right)\right)}_{\text{Risk premium on dollar } n \text{ inflation}}
\end{aligned}$$

Proof. See appendix B.1. □

Assumption 4 implies

$$\text{Cov}_t\left(\frac{\xi_{t+j}}{\mathbb{E}_t\left[\xi_{t+j}\right]}, \frac{e_{t+j}^{(n)}/e_t^{(n)}}{\mathbb{E}_t\left[e_{t+j}^{(n)}/e_t^{(n)}\right]}\right) = \text{Cov}_t\left(\frac{S_{t+j}/S_t}{\mathbb{E}_t\left[S_{t+j}/S_t\right]}, \frac{e_{t+j}^{(n)}/e_t^{(n)}}{\mathbb{E}_t\left[e_{t+j}^{(n)}/e_t^{(n)}\right]}\right) = 0,$$

so Lemma 1 highlights that the assumption turns off both a risk premium on dollar n inflation and exposures to comovements between ξ_{t+j} and $e_{t+j}^{(n)}$. It retains a risk premium on haircut risk.

Assumption 4's stated independence between S_t and $\{e_t^{(n)} : n \in \{g, d\}\}$ has more than one possible economic interpretation. One is that the economy satisfies a classical real-nominal dichotomy and that the money supply does not feedback on real variables. In this case, real state variables determine a stochastic discount factor process that evolves independently of nominal prices. The risk premium on dollar n inflation disappears because there is no risk to price. An alternative interpretation might be that bond holders are risk neutral and so do not price dollar n inflation risk. Such a risk neutrality assumption is too confining for us because it also eliminates a risk premium on haircut risk, a feature we want to retain when in subsection 6.4 we compare yields on US and UK debt.

The assumption of conditional independence between ξ_t and $\{e_t^{(n)} : n \in \{g, d\}\}$ succeeds in decomposing default risks into two orthogonal components: a haircut risk, ξ_{t+j}, and a currency risk, $e_{t+j}^{(n)}/e_t^{(n)}$. In appendix B.2, we discuss how to relax this conditional independence assumption and allow for "haircuts" through denomination change.

4.5 Implementing Our Pricing Theory

Suppose that at time t we observe prices on an integer number M_t of coupon-bearing government bonds. To reflect the diverse bonds in our sample, we allow a given bond, i, to promise of either: (i) a sequence of gold dollar coupon and principal payments $\{\overline{m}_{t+j}^{(i,g)}\}_{j=1}^{\infty}$, or (ii) a sequence of greenback dollar coupon and principal payments $\{\overline{m}_{t+j}^{(i,d)}\}_{j=1}^{\infty}$. We allow $\overline{m}_{t+j}^{(i,n)}$ to be zero, where $n \in \{g, d\}$ indicates two types of dollar denominations. Most bonds have finite maturities so we let $J_t^{(i)}$ denote the remaining number of periods with non-zero payments.[16] As before, let $p_t^{(i,n)}$ denote the price of a type n dollar bond in terms of gold, where $n \in \{g, d\}$. To account for differences in maturities and coupons, we view each coupon-bearing bond i as a basket of zero-coupon securities and use pricing formulas:

$$p_t^{(i,n)} = \begin{cases} \sum_{j=1}^{\infty} q_t^{(j,g)} \overline{m}_{t+j}^{(i,g)}, & \text{if } n = g \\ \sum_{j=1}^{\infty} q_t^{(j,g)} z_t^{(j)} \overline{m}_{t+j}^{(i,d)}, & \text{if } n = d \end{cases} \tag{4.5}$$

To prepare the way for expressing asset prices as inner products of price and quantity vectors, let:

- $\mathsf{q}_t := \{q_t^{(j,g)}\}_{j=1}^{\infty}$ denote a sequence of gold dollar zero-coupon bond prices
- $\mathsf{z}_t := \{z_t^{(j)}\}_{j=1}^{\infty}$ denote a sequence of greenback denomination adjustment factors
- $\overline{\mathsf{m}}_t^{(i,n)} := \{\overline{m}_{t+j}^{(i,n)}\}_{j=1}^{\infty}$ denote a sequence of promised *coupon and principal* payments in currency n where $n \in \{g, d, a\}$ and it is understood that $\overline{m}_{t+j}^{(i,n)} = 0$ for $j > J_t^{(i)}$.

[16] In case of perpetual *consols*, $J_t^{(i)} = \infty$.

We can then write our pricing formulas as the inner products:

$$p_t^{(i,g)} = \left\langle \mathsf{q}_t,\ \overline{\mathsf{m}}_t^{(i,g)} \right\rangle \qquad \text{gold dollar bonds}$$

$$p_t^{(i,d)} = \left\langle \mathsf{q}_t \odot \mathsf{z}_t,\ \overline{\mathsf{m}}_t^{(i,d)} \right\rangle \qquad \text{greenback bonds}$$

where $\langle\cdot\rangle$ denotes an inner product (on the space of real sequences) and $\odot$ denotes a Hadamard (element-wise) product. If we explicitly express q_t and z_t in their parametric forms, then this becomes:

$$p_t^{(i,g)} = \left\langle \mathsf{q}(\tilde{\beta}_t),\ \overline{\mathsf{m}}_t^{(i,g)} \right\rangle \qquad \text{gold dollar bonds}$$

$$p_t^{(i,d)} = \left\langle \mathsf{q}(\tilde{\beta}_t) \odot \mathsf{z}(\zeta_t),\ \overline{\mathsf{m}}_t^{(i,d)} \right\rangle \qquad \text{greenback bonds}$$

Observe that assumptions 3 and 5 parameterize q_t and z_t. This means that we can use information about $p_t^{(i,n)}$, $\overline{m}_t^{(i,n)}$, and $e_t^{(n)}$ for $n \in \{g, d, a\}$ together with equations (4.2), (4.3), and (4.5) to infer parameters, $\tilde{\beta}_t$ and ζ_t, that pin down our yield curves for greenback dollar and gold dollar denominated zero-coupon bonds. We elaborate in the next section.

5 Econometric Strategy

Using our pricing formulas from section 4.5 we build a statistical model for which a likelihood function can be derived. To this end, we introduce three types of Gaussian shocks: (1) shocks that move parameter vectors β_t and ζ_t over time, (2) pricing errors for each bond, and (3) forecast errors for the model of exchange rates. These building blocks give rise to a richly parametrized—yet tractable—non-linear state space model. To estimate its (more than $7,500$) parameters, we apply Bayesian Markov Chain Monte Carlo (MCMC) methods. We specify weakly informative prior distributions for the model's hyper-parameters (see Appendix C) with the specific purpose of *regularizing* our estimator and facilitating smooth operation of the sampling algorithm. Importantly, we do not aim to choose these priors to faithfully summarize our subjective beliefs, rather we view them as tools that help our statistical model make reliable inference about the objects we care about.

5.1 Pooling Across Time

Assumptions 3 and 5 explicitly make the yield curve parameters β_t and (some) parameters of the exchange rate dynamics ζ_t time-dependent, which raises the question: how the different components relate to each other over time? A widely used yield curve estimate—available for the period since 1960—by Gürkaynak et al. (2007) assumes no intertemporal dependence among the four parameters in (4.2): they estimate a different set of βs and τ for each t using only bond prices available at date t. Therefore, nothing learned about the yield curve any one date contributes to their estimates for other dates no matter how close. Unlike the post-WW2 period, prior to the First World War, price data are sparse and coverage varies across time. Consequently, we in effect need to pool information over time to estimate a time series of yield curve parameters. To this end, we introduce a multilevel (a.k.a. an hierarchical) statistical model.

Gold denominated yield curve: At any given t, the shape of the gold denominated yield curve is summarized by four parameters: the three time-varying parameters in the β_t vector and the time-invariant τ. Diebold and Li (2006) show that the β parameters can be interpreted as latent "level", "slope" and "curvature" factors of the yield curve. We model these factors as random walks:

Assumption 6. Parameter vector β_t is a drift-less random walk:

$$\beta_t = \beta_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_{\beta,t} \qquad\qquad \varepsilon_{\beta,t} \sim \mathcal{N}(\mathbf{0}, \mathbb{I}_3), \ \forall t \geq 1.$$

Parameter matrix Σ governs how evidence about a yield curve at one date affects inferences about yield curves at other dates. The closer are two dates to each other, the more correlated are the associated yield curves, with Σ capturing what "close" means. The case $\Sigma \to 0$ corresponds to *complete pooling*: here the yield curve is assumed to be fixed over time so that each observation has an equal influence as all other dates. Contrary situations in which $\Sigma \to \infty$ call for *no pooling*: there is no relationship between adjacent parameter estimates, we use only period t information to estimate period t yield curve parameters as in Gürkaynak et al. (2007). By inferring Σ from the data, we learn how much pooling across time we should do to improve estimates in light of intertemporal imbalances in data availability.

We allow shocks to different components of β_t to be correlated so that Σ need not be a diagonal matrix. This enables us to infer relatively precise estimates of the short end of the yield curve throughout the whole sample period. Indeed, the bottom panel of Figure 2 shows big gaps in the maturity structure for certain sub-periods. In particular, the early decades of the 19th and 20th centuries are characterized by relatively few short-term outstanding bonds. Assuming that different parts of the yield curve follow correlated but time-invariant dynamics allows us to transmit what we learn about co-movements between short- and long-term yields from years when many maturities are outstanding (as in the second half of the 19th century) to years when data about short-term yields are scarce.

For numerical stability, we model correlation coefficients and variances of marginal distributions of shocks to β_t separately by decomposing covariance matrix Σ as[17]

$$\Sigma = \Xi \Omega \Xi, \tag{5.1}$$

where Ω is the correlation matrix and Ξ is a diagonal matrix containing the marginal standard deviations σ_i of the λ_t shocks. This decomposition implies $\sigma_i := \Xi_{i,i} = \sqrt{\Sigma_{i,i}}$ and $\Omega_{i,j} = \frac{\Sigma_{i,j}}{\sigma_i \sigma_j}$ for $i, j \in \{1, 2, 3\}$.

Prior (on gold denominated yield curve): Assumptions 3 and 6 give rise to a flexible model of the gold denominated yield curve process that is pinned down by a small set of hyper-parameters. We specify a prior on τ and the initial (time 0) β vector that effectively determines an "average yield curve" for the whole sample period. We use log-normal prior for τ and independent Gaussian priors for the three entries of the initial β vector that implies the prior distribution for the initial yield curve shown in the left panel of Figure 3. Our prior imposes a flat "average yield curve", i.e., for all maturity the prior mean is 10% with standard deviation of around 10%.

While the prior distribution tends to concentrate around this yield curve in the sense that the prior *mean* of $y_t^{(j)}$ is independent of t $\forall j \geq 0$, the random walk specification of β_t implies that the prior *variance* grows linearly with time. Consequently, while the "average yield curve" influences our posterior distribution in the early part of the sample, it is much less influential later. The right panel of Figure 3 illustrates how "prior coverage bands" for the 10-year yield grow over time. How much our prior for β_0 affects the posterior distribution for later periods depends mainly on our prior on Σ. We use weakly informative priors for components of Σ:

- For the standard deviations we use a *common* exponential prior (independent across components) with the rate parameter tuned so that *a priori* the probability that $\sigma_i > 1$ is less than 5%. The mean is 1/3.

[17] See Barnard et al. (2000).

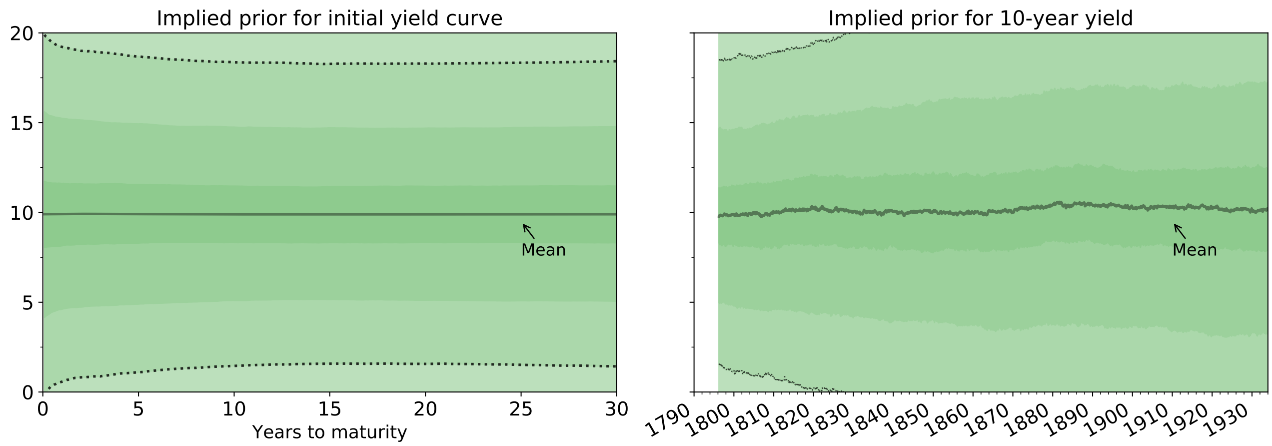

Figure 3: Implied prior distribution of the initial yield curve and the 10-year zero-coupon yield.

The solid grey lines depict the mean, dotted lines depict the 25% and 75% percentiles of the prior distribution. Shaded areas represent interquantile ranges so that dark areas are indicative of concentrated prior probability.

- For the correlation matrix Ω we use the LKJ prior with a concentration parameter $\eta = 5$, which is a unimodal but fairly vague distribution over the space of correlation matrices. For η values larger than 1, the LKJ density increasingly concentrates mass around the unit matrix, i.e., favoring less correlation.[18]

Model of exchange rates: Approximation of greenback denominated yield curves for the period between 1862 and 1879 requires us to infer a new object from data: the multiplier $z_t^{(j)}$ defined in (4.3) that Assumption 5 models with the use of a state-space model. As we discussed in section 4.3, we allow the long-run mean μ_t and persistence A_t parameters of this state-space model move over time. Similar to β_t we assume that they follow drift-less random walks, but in order to economize on the number of parameters (and to avoid over-fitting the few greenback denominated bonds we observe) we make the parameter changes infrequent.

Assumption 7. Parameters μ_t and A_t follow drift-less random walks with shocks that arrive every Δ months:

$$\mu_t = \begin{cases} \mu_{t-1} + \Sigma_\mu \varepsilon_{\mu,t} & \varepsilon_{\mu,t} \sim \mathcal{N}(\mathbf{0}, \mathbb{I}_2) \quad \text{if } \; t = k\Delta \quad \text{for } k \in \mathbb{N} \\ \mu_{t-1} & \text{otherwise} \end{cases}$$

$$\text{vec}(A_t) = \begin{cases} \text{vec}(A_{t-1}) + \Sigma_A \varepsilon_{A,t} & \varepsilon_{A,t} \sim \mathcal{N}(\mathbf{0}, \mathbb{I}_4) \quad \text{if } \; t = k\Delta \quad \text{for } k \in \mathbb{N} \\ \text{vec}(A_{t-1}) & \text{otherwise} \end{cases}.$$

As $\Delta \to \infty$, the frequency of parameter updates goes to zero, providing a linear state-space model (4.4) with time-invariant long-run mean μ and persistence A.[19] An advantage of this formulation is that while it nests a first-order vector autoregressive process (VAR), it is more flexible due to the presence of the latent variables x_t that allow for first-order moving-average dynamics. By setting $\Delta \geq 1$ to low values, we let the long-run mean and persistence of v_t move over time. This provides a simple way to introduce time-variation in the amount of information pooling: we divide the period of interest into subperiods of equal length Δ and assume complete pooling within subperiods and partial pooling—parameterized by Σ_μ and Σ_A—across subperiods.

[18] See Lewandowski et al. (2009). The LKJ distribution is defined by $p(\Omega|\eta) \propto \det(\Omega)^{\eta-1}$. For $\eta = 1$, this is a uniform distribution.

[19] This fixed parameter model has been used by Piazzesi and Schneider (2007) and Szőke (2021) (among others) to estimate conditional moments of inflation and consumption growth in consumption-based asset pricing models.

5.2 Bond-specific Pricing Errors

Apart from the obvious difficulties arising from transcribing and time aggregating price quotations from newspapers, we have other reasons to believe that, in certain periods, particular bonds violate the collection of subsection 2.4 assumptions that support our asset pricing equations. We address this issue by introducing *bond specific* pricing errors, modelled as random variables with Gaussian distribution (in assumption 8) and by using our statistical model to estimate error distribution parameters jointly with yield curve parameters.

Assumption 8. Each bond has a pricing error with the following stochastic properties: errors on bond i are independent from errors on other bonds and the distribution of errors on bond i is a time-invariant Gaussian distribution with mean 0 and standard deviation $\sigma_m^{(i)}$.

Introducing these errors enables our statistical model to decrease the influence of peculiar bonds on our yield estimates. We view the imposition of non-bond specific haircut risk in assumption 2 as our key identification assumption. If assumption 2 were violated and some bonds had idiosyncratic $\xi^{(i)}$ processes, then we would expect to see large estimates of $\sigma_m^{(i)}$ for some of the observable bonds. We use this as a diagnostic tool to inspect whether we should change the way we treat the cash flows from particular bonds, exclude particular bonds from the estimation of certain yield curves, or divide bonds into new subgroups according to their common characteristics and re-estimate a yield curve for each group separately. If yield curve estimates differ across the subgroups, then we can interpret the difference as the yield premium that arises from the specific characteristic.

Example: Ambiguous denomination. In our first estimation of the gold denominated yield curve, we included the five-twenties as bonds that definitely paid principals in gold even though their contract left the denomination ambiguous. Under this assumption, we found that bond specific pricing errors were very high for bonds traded during the Civil War period. This prompted us to treat the bonds that had ambiguous denomination separately and look for additional price data.

Example: Potential convenience yields. Idiosyncratic variations in how easily bonds could be used for transactions would lead to large estimates of $\sigma_m^{(i)}$. We believe this occurred during the War of 1812, when the US issued a collection of short term Treasury notes that were used for payments well after their earliest redemption date.[20] Initially, we included these bonds in the estimation of the gold yield curve but we found large pricing errors during the War of 1812, so we ended up dropping them in an effort to restrict our sample to non-money Federal liabilities.

Aside: Time-varying pricing errors. An alternative pricing error specification would assign the same $\sigma_{m,t}$ to all bonds available in period t. We refer to this as a time-varying pricing error model. Unlike our bond-specific pricing error model, this specification equalizes influences of different bonds on yield curve estimates—the likelihood function does not let the yield curve price a subset of bonds well at the cost of large pricing errors for other bonds. Large $\sigma_{m,t}$ estimates can be interpreted as indicating that our common payment-risk assumption (Assumption 2) is violated for the corresponding subperiod.

[20] Bayley (1882) suggests that such notes included the Treasury Notes of 1812, Treasury Notes of 1813, Treasury Notes of March 1814, Treasury Notes of December 1814, and the Small Treasury Notes of 1815

5.3 A Nonlinear State Space Model of Bond Prices

Using these building blocks, we can write our nonlinear state space model in the following compact forms:

$$\begin{aligned}
p_t^{(i)} &= \left\langle \mathsf{q}(\tilde{\beta}_t),\ \overline{\mathsf{m}}_t^{(i,g)} \right\rangle + \sigma_m^{(i)} \varepsilon_t^{(i)} && \text{gold bonds} \\
p_t^{(i)} &= \left\langle \mathsf{q}(\tilde{\beta}_t) \odot \mathsf{z}(\zeta_t),\ \overline{\mathsf{m}}_t^{(i,d)} \right\rangle + \sigma_m^{(i)} \varepsilon_t^{(i)} && \text{greenback bonds} \\
\beta_t &= \beta_{t-1} + \Sigma^{\frac{1}{2}} \varepsilon_{\beta,t} && \text{yield curve parameters} \\
\zeta_t & \quad \text{see Assumption 7} && \text{expectation parameters} \\
\text{with} \quad & \varepsilon_t^{(i)} \sim \mathcal{N}(0,1) \ \ \forall i, \forall t \geq 1 \qquad \varepsilon_{\beta,t} \sim \mathcal{N}\left(\mathbf{0}, \mathbb{I}_3\right),\ \forall t \geq 1
\end{aligned}$$

where $p_t^{(i)}$ denotes the *observed* period-t price of bond i in terms of gold. The posterior distribution of this model is obtained by adding up the Gaussian log-likelihoods associated with the independent shocks and combine them with priors described in Appendix C. This model has four types of parameters that we need to estimate: (1) yield curve parameters $\{\beta_t\}_{t=1796:1}^{1933:12}$ and τ, (2) expectation parameters $\{\zeta_t\}_{t=1862:1}^{1878:12}$, (3) measurement errors $\{\sigma_m^{(i)}\}_{i=1}^{264}$, and (4) "smoothing parameters" Σ, Σ_μ, and Σ_A.

This model has approximately $7,500$ parameters. We cope with this high-dimensional parameter space by using Bayesian methods, namely, Hamiltonian Monte Carlo with a "No-U-Turn Sampler" (HMC-NUTS) of Hoffman and Gelman (2014), along with subsequent developments described in Betancourt (2018). The basic idea of the method is to use slope information about the log-likelihood to devise an efficient Markov Chain Monte Carlo sampler. This method can attain a nearly i.i.d. sample from the posterior by proposing moves to distant points in the parameter space through (an approximately) energy conserving simulated Hamiltonian dynamic. While it has been used extensively in statistics, economic applications are relatively rare to date.

Computational issues: While `Stan` might seem an obvious choice for the task at hand—it is a well-developed software that provides an efficient implementation of the HMC-NUTS sampler—non-trivial features of our data set make it inconvenient for our purposes. Some of the main technical difficulties we face are: (1) the number of observed assets change over time, (2) each bond has a payoff stream of varying length, (3) periods without price observations, (4) the set of bond-specific pricing errors that are relevant at a given period t changes over time in complicated fashion, (5) we want to estimate exchange rate expectations only for a specific sub-period, etc. To tackle these difficulties, we code the log posterior function of our model from scratch in `Julia` (Bezanson et al., 2017) and feed it into the DynamicHMC.jl package by Papp et al. (2021) which is a robust implementation of the HMC-NUTS sampler mimicking many aspects of `Stan`. An important advantage of this package is that it allows the user to provide the Jacobian of the log-posterior manually. Not having to rely on automatic differentiation for a model with $7,500+$ parameters cuts running time by an order of magnitude. In most cases, we use the recommended (default) parameters for the NUTS algorithm.

6 Gold Dollar Yield Curves: 1790-1933

We now describe salient features of our approximated gold denominated yield curves for the gold standard period from 1790-1933. We start with 10-year yields since the economic history literature often focuses on this maturity. Because a maturity neighborhood of 10 years is arguably where the best estimates in the literature reside, studying 10-year yields provides sensible check on our results. We then discuss our approximation for historical 1-year yields and for spreads between short and long term yields.

6.1 Yields on 10-Year Zero Coupon Bonds

Figure 4 depicts our estimates of 10-year gold denominated zero-coupon yields along with the "Federal Government Bonds: Selected Market Yields" series of Homer and Sylla (2004). Homer and Sylla (2004) computed their US long term market yield series as the coupon rate on US federal bonds that have approximately 10 years to maturity and were trading close to par.

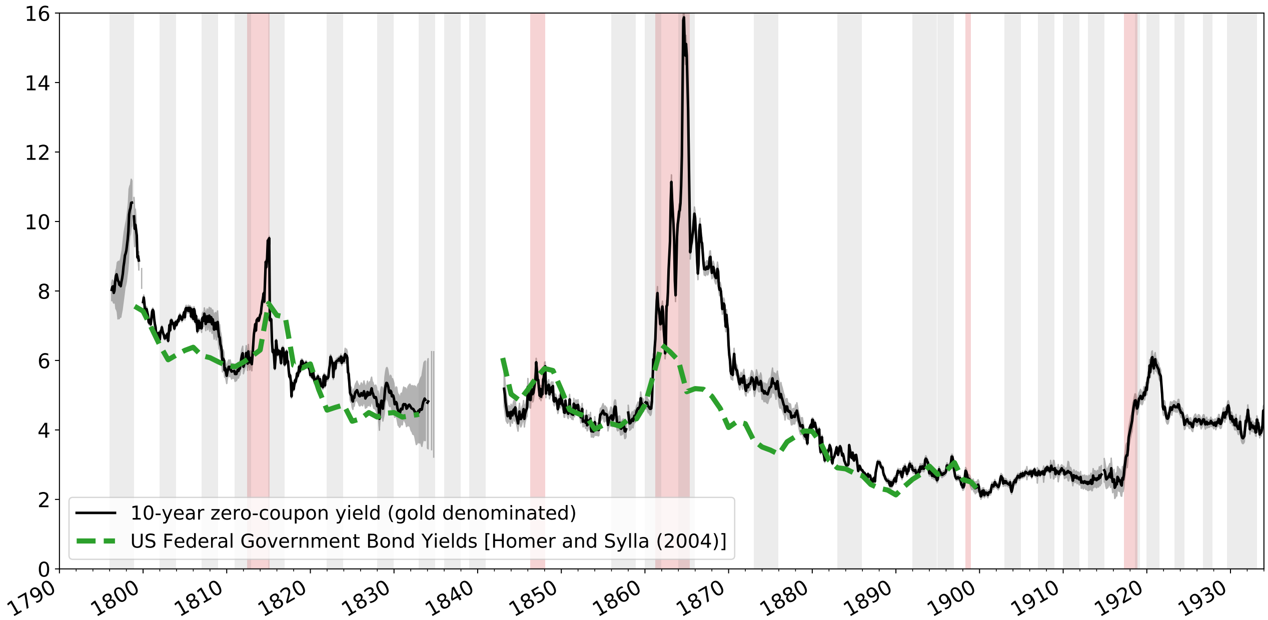

Figure 4: Long-Term Yield Estimates.

The solid black line depicts the mean of our posterior estimate for the 10-year, gold denominated, zero coupon yield. The dashed grey line depicts the mean of our posterior estimate for the 10-year, dollar denominated, zero coupon yield. The grey bands around the posterior mean depict the 95% interquantile range. The dashed green line depicts the 'Federal Government Bonds: Selected Market Yields' series from Table 38 of Homer and Sylla (2004). The light gray intervals depict recessions as dated by Davis (2006) for the 1796-1914 period and NBER recessions thereafter. The light red intervals depict wars (from left to right: the War of 1812, the Mexican-American War, the Civil War, the Spanish-American War, and World War I).

Evidently, our estimates typically follow the Homer and Sylla (2004) series, except that we estimate substantially higher yields during the War of 1812 and the Civil War. We find it reassuring that our estimate aligns with Homer and Sylla (2004) during "non-emergency" periods because there are good reasons to think that their estimates should be a good approximation to the 10 year yield. Their approach calculates an average yield to maturity for 10 year bonds, which should be similar to the 10 year zero-coupon yield when the yield curve is relatively flat.[21] Except during and after the Civil War, the average duration of outstanding bonds was close to 10 years and the average market trading price is close to par and Homer and Sylla (2004) have a large data set.[22] For these reasons, we consider the general congruence between our estimated 10-year yields and 'long-term federal government bond yields' in Homer and Sylla (2004) as a reassuring check on the plausibility of our findings.

Our approximating yields are quite volatile during the 1790s when secondary markets in Treasury securities were thin. Yields fell steadily from January 1791 to March 1792 when a financial panic caused sharp drops in bond prices and corresponding increases in yields. Ten-year zero-coupon yields remained high for the remainder

[21] We discuss the relationship between the zero-coupon yield curve and the yield to maturity in appendix B.3.
[22] Bonds typically traded close to par because the government set coupon rates to ensure an issue price of par.

of the decade and spiked at 12.3% in August 1798, one month after the Congress authorized a 15-year loan paying an 8% coupon to cover increased military spending at the outbreak of the Quasi-War with France. Yields trended downward thereafter, and by 1803 the US government was able to issue at par a $11.25 million 15-year loan with a 6% percent coupon to finance the Louisiana Purchase.

An advantage of our approach is that we can approximate 10-year yields not only during "non-emergency" periods but also during the War of 1812 and Civil War, when prices were volatile and deviated substantially from par. As reported in figure 4, during the War of 1812, the 10-year zero-coupon yield spiked to over 9 percent. A big source of funds for this war was the Treasury's issuing of five long-term loans with face values totaling $66 million. Resistance to the war mainly from Federalists in the Northeast and a failure to replace lost customs revenue with internal taxes forced the Treasury to sell these bonds at deep discounts. Bayley (1882) reports that two of these loans were sold at 12% discounts, and a third was sold at a 20% discount. Those officially-stated discounts understate the true discounts, since for payment the Treasury actually accepted at face value bank notes whose market values had sunk substantially below par.

The Treasury again had trouble selling new bonds at par during the Civil War, leading to much higher yields.[23] Our 10-year yield estimate reaches a peak of 16% near the end of the Civil War, which is substantially higher than the Homer and Sylla (2004) series peak of 6% at the start of the war. The following observations suggest that our estimate of yields during this period is more plausible than those of Homer and Sylla (2004). Starting in 1862, all US Treasury bonds could be purchased with greenback dollars, including bonds with coupons and principal payments being denominated in diverse units of account, some in greenbacks, others in gold dollars. The value of the greenback fluctuated with battlefield and political news, and all Treasury bond prices deviated substantially from par. For example, during the summer of 1864, when re-election of President Abraham Lincoln was in doubt, 100 greenback dollars could be purchased for as few as 40 gold dollars. Consequently, during that time Treasury bonds that promised to pay 6 percent coupons in gold dollars could be purchased for 40 percent of par, implying long-term yields in excess of 15 percent.

The Homer and Sylla (2004) series depicted in figure 4 is not the long-term US bond series that is commonly used in the economic history literature. Instead, researchers[24] have typically used a 'composite series' that combines the Homer and Sylla (2004) estimates for the period from 1798-1861 with the yield-to-maturity on the New England Municipal bond for the period 1862-1899 and the yield-to-maturity on corporate bonds for the period 1900-1940.[25] Figure 5 plots this composite series alongside our 10-year yields. Our estimates diverge post 1861 when the composite series stops using US federal debt prices. We estimate a much higher long-term yield during the war and a lower long-term yield in the late nineteenth century. Possible sources for these discrepancies are that federal debt carried a greater default risk during the Civil War and that, after the war, National Banking Era protocols stimulated demands for federal bonds as reserves against National Bank Notes.

[23] Homer and Sylla (2004) themselves caution against using their estimates for the Civil War period stating on page 303, "...the tables of bond yields for the years 1863 to 1870 do not provide a reliable picture of long-term interest rates." This is because there were no federal bonds trading with a gold price of par and so they are forced to estimate the yield as the gold coupon rate for bonds trading with a greenback price of par. We can capture greater variation in the yield curve because we use the universe of US Treasury bonds at monthly frequency whereas Homer and Sylla (2004) use the subset of these bonds that are trading at par.

[24] For example, Officer and Williamson (2021), Shiller (2015), Jordà et al. (2019), and Hamilton et al. (2016).

[25] It is not obvious that during the 19th century municipal debt was a safer investment than federal debt. Until the 1934 Gold Reserve Act, the federal government had never defaulted. In contrast, eight states and one territory defaulted in 1830s and 1840s and ten states defaulted in 1870s and 1880s. These state defaults are discussed in McGrane (1935) and English (1996).

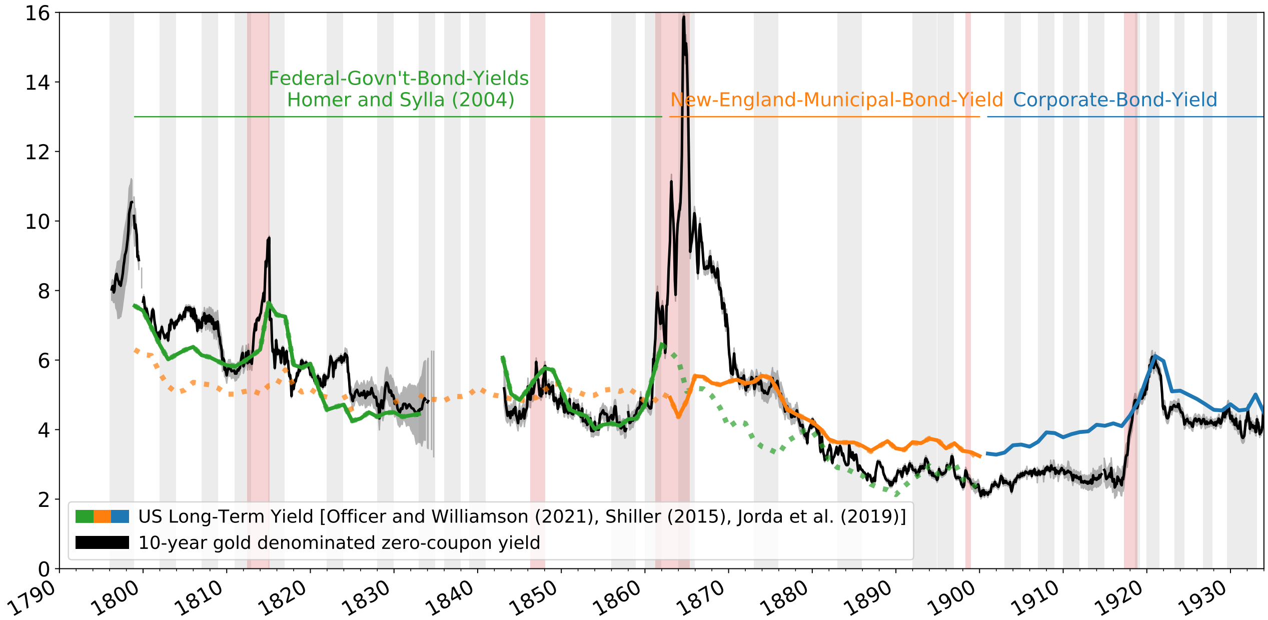

Figure 5: Alternative Long-Term Yield Estimates.

The solid black line depicts the mean of our posterior estimate for the 10-year, gold denominated, zero coupon yield. The dashed grey line depicts the mean of our posterior estimate for the 10-year, dollar denominated, zero coupon yield. The grey bands around the posterior mean depict the 95% interquantile range. The green line (bold and dotted) depicts the 'US Government Bond Yield' series from Homer and Sylla (2004). The orange line (bold and dotted) depicts the New England Municipal Bond Yield reported by Homer and Sylla (2004). The blue line depicts the Corporate Bond Yield reported by Homer and Sylla (2004). The bold green-orange-blue line depicts the 'composite' bond series used by Officer and Williamson (2021). The light gray intervals depict recessions as dated by Davis (2006) for the 1796-1914 period and NBER recessions thereafter. The light red intervals depict wars (from left to right: the War of 1812, the Mexican-American War, the Civil War, the Spanish-American War, and World War I).

6.2 Short Term Yields

Figure 6 depicts our estimates for 1-year gold denominated zero-coupon yields alongside a short term yield series used by Officer and Williamson (2021) and Jordà et al. (2019).[26] We have more difficulty estimating the 1-year yields than the 10-year yields because some periods have very few price observations for bonds that are close to maturity. This is reflected in sizes of 95% interquantile ranges for 1-year zero-coupon yields in figure 6. We are most concerned about the period 1790-1815 when our only price observations are for the consol bonds that Alexander Hamilton issued to refinance the Revolutionary War debts.[27] By contract, the Hamilton consols had no maturity dates. Because the Federal government ended up repurchasing and retiring all of these bonds, our perfect foresight assumption means that we treat them as finite maturity bonds.[28] This allows us to estimate a yield curve, but we are faced with two problems: investors may not have anticipated that the bonds would be repurchased and when, and "times-to-repurchase" were typically greater than 10 years, providing us with little information about the short end of the yield curve. For these reasons, we drop data from 1790-95 and treat the

[26] The figure depicts the series labeled as "Surplus Funds (Contemporary Series)." The Series involves the short-term lending or borrowing of surplus funds, that is, funds that are considered excess by the lending institution and are required for immediate temporary use by the borrowing entity.

[27] Bayley (1882) calls these bonds: *The Six Percent Stock of 1790*, *The Deferred Six Percent Stock of 1790*, and *The Three Percent Stock of 1790*.

[28] The time to maturity in figure 2 shows the time until the bonds were bought back by the government. The Act authorizing the issuance of the 1790 Stocks provided for a committee comprised of the president of the Senate, Chief Justice, Secretary of State, Secretary of the Treasury, and Attorney General to use surplus revenue to repurchase these stocks at market prices, if not exceeding par. Between 1791 and 1824, nearly all of the outstanding Six Percent and Deferred Six Percent Stocks were repurchased. By 1832, nearly all of the outstanding Three Percent Stock was repurchased. See Bayley (1882, pages 33, 110).

short yield curve during 1790-1815 with caution.

Our short term yield series substantially departs from popular alternative series, especially during the Civil War when we estimate substantially higher yields, peaking at approximately 44% in July 1864. Anecdotal evidence indicates that Union short-term debt paid very high yields during the Civil War. For example, Homer and Sylla (2004, page 302) report that in 1860 the Treasury had issued one-year notes at rates of 10-12% and had rejected bids ranging from 15-36%. One-year yields are negative in the early 1880s and close to zero in the early 1890s. What parts of our data most influence our inferences about these negative yields? It is that these negative yields help price both the *Four Percent Loan of 1907* and the *Four and One-Half Percent Loan of 1891*.[29] Economic events that may or may not be sources of these low gold yields during the early 1880s are that financial markets were highly volatile, that the US government was using surpluses to repurchase bonds, and that the US had just returned the gold standard in January 1879 (see Noyes, 1909, pp. 79-80).

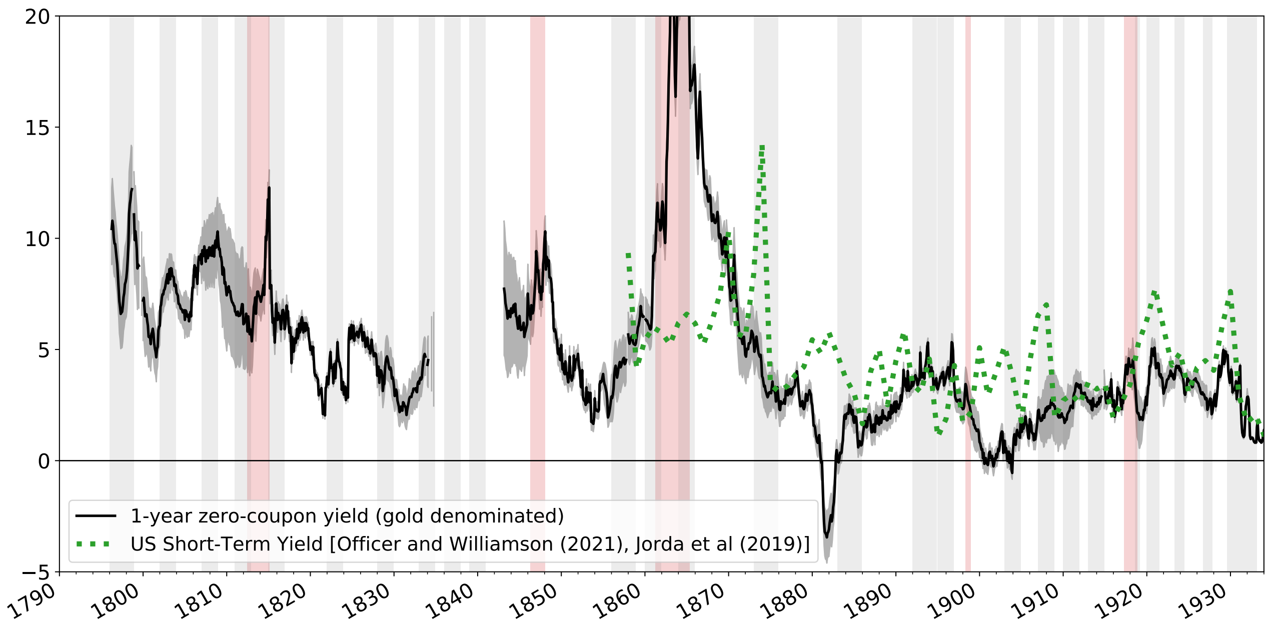

Figure 6: Short-Term Yields.

The solid black line depicts the mean of our posterior estimate for the 1-year, gold denominated, zero coupon yield. The dashed grey line depicts the mean of our posterior estimate for the 10-year, dollar denominated, zero coupon yield. The grey bands around the posterior mean depict the 95% interquantile range. The green dotted line depicts the US short term yield series (surplus funds, contemporary) used by Officer and Williamson (2021) and Jordà et al. (2019). The light gray intervals depict recessions as dated by Davis (2006) for the 1796-1914 period and NBER recessions thereafter. The light red intervals depict wars (from left to right: the War of 1812, the Mexican-American War, the Civil War, the Spanish-American War, and World War I).

6.3 Term Spreads

Figure 7 depicts the yield on 5-year government bonds minus the yield on 1-year government bonds. We refer to this as a term spread. A positive term spread indicates an upward sloping yield curve (i.e., longer maturity bonds have higher rates), while a negative term spread indicates an inverted yield curve (i.e., shorter maturity

[29] These are the names used in Bayley (1882). We initially imposed non-negativity constraints in the estimate of the yield curve. This led to small pricing errors for the *Four Percent Loan of 1907* but large pricing errors for the *Four and One-Half Percent Loan of 1891* in the early years of the 1880s. Relaxing the non-negativity constraint significantly reduced the pricing errors on the *Four Percent Loan of 1907* without increasing other errors. We take this as suggestive statistical evidence that the yield curve went negative in the early 1880s, but further investigation is required.

bonds have higher rates). Yield curves were typically upward sloping throughout the nineteenth century, with notable inversions during the War of 1812, the early 1830s, the Mexican-American War, the Civil War, and in the late 1890s.

A large literature has used yields to help predict real GDP growth.[30] Our yield curve estimates open the way to extend such work back into the nineteenth century. As a preliminary step, our table 1 below emulates table 2 from Ang et al. (2006). It reports the coefficient $\beta_k^{(j)}$ and R^2 for the regression:

$$g_{t+k} = \alpha_k^{(j)} + \beta_k^{(j)} \left(y_t^{(10)} - y_t^{(j)} \right) + \varepsilon_{t+k,k}^{(j)}$$

where g_{t+k} is the annual percentage growth of real GDP over the next k years and $y_t^{(j)}$ denotes the annualized j-year zero coupon yield for $j \in \{1, 5\}$. Notice that an upward sloping yield curve appears to be positively correlated with future economic growth during the 19th century even though no central bank existed to engage in "active" monetary policy.[31] In table 2 in appendix E, we report the coefficients from the regression of the change in the spread on GDP growth and find additional suggestive evidence that nineteenth century spreads have some predictive ability.

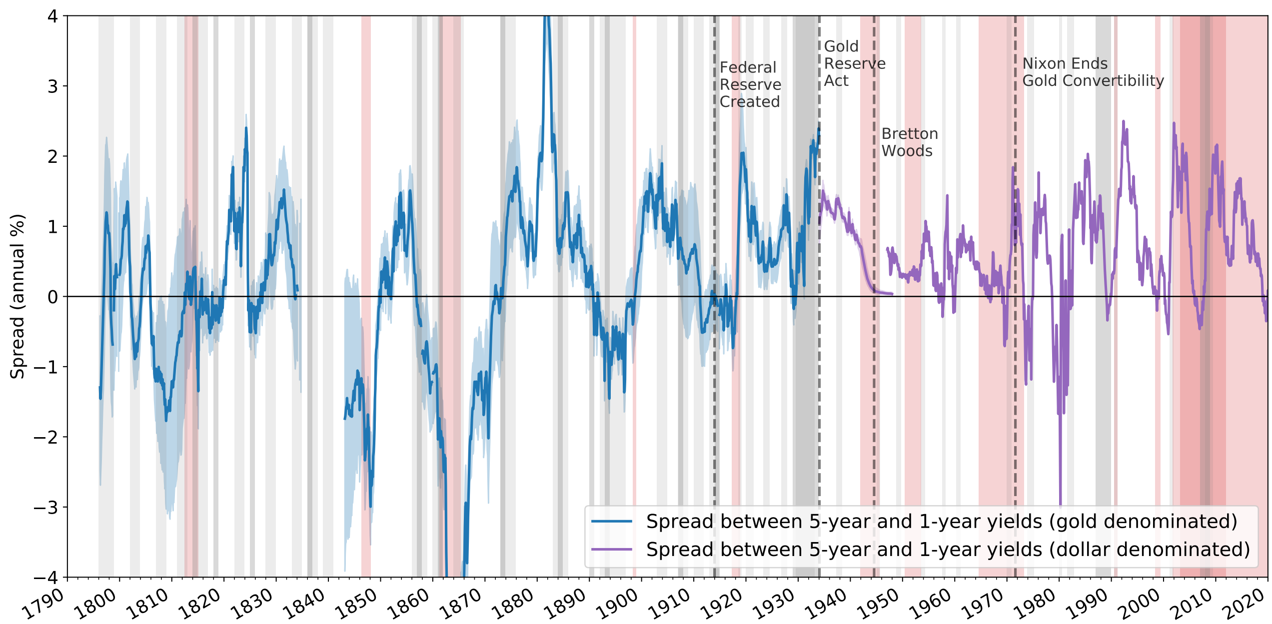

Figure 7: 5 Year – 1 Year Yield Spread

The solid blue line depicts the yield on 5-year, gold denominated, zero coupon US government bonds minus the yield on 1-year, gold denominated, zero coupon US government bonds. The pale blue bands around the posterior mean depict the 95% interquantile range. The purple line depicts the same yield spread for dollar denominated bonds (after the US leaves the gold standard). The light gray intervals depict recessions as dated by Davis (2006) for the 1796-1914 period and NBER recessions thereafter. The dark gray intervals depict NBER recessions. The light red intervals depict wars (from left to right: the War of 1812, the Mexican-American War, the Civil War, the Spanish-American War, and World War I).

[30] See Stock and Watson (2003) for a critical literature review.

[31] However, from 1897 until 1913, Republican Secretaries of the Treasury more and more violated the letter of the 1844 Independent Treasury Act by *de facto* conducting open market operations intended to lean against the wind.

	1797-1860				1866-1933				1950-2000			
	Term spread maturity											
Horizon	10y - 1y		10y - 5y		10y - 1y		10y - 5y		10y - 1y		10y - 5y	
k-years	β_k^1	R^2	β_k^5	R^2	β_k^1	R^2	β_k^5	R^2	β_k^1	R^2	β_k^5	R^2
1-year	0.29	0.028	0.95	0.042	0.07	0.000	0.43	0.002	1.28	0.230	3.54	0.159
	(0.25)		(0.68)		(0.28)		(0.69)		(0.46)		(1.66)	
3-year	-0.32	0.006	-0.01	0.000	0.27	0.002	1.35	0.006	1.56	0.103	2.71	0.028
	(0.49)		(1.30)		(0.69)		(1.84)		(0.72)		(1.13)	

Table 1: Forecasts of real GDP growth from term spreads

The table reports the coefficient $\beta_k^{(j)}$ and R^2 for the regression $g_{t+k} = \alpha_k^{(j)} + \beta_k^{(j)} \left(y_t^{(10)} - y_t^{(j)} \right) + \varepsilon_{t+k,k}^{(j)}$ where g_{t+k} is the annual percentage growth of real GDP over the next k years and $y_t^{(j)}$ denotes the annualized j-year zero coupon yield. We annualize the yields by taking the arithmetic average for each year. Newey and West heteroskedasticity- and autocorrelation-consistent standard errors with lag order one in parentheses. *** 1%, ** 5%, and * 10% significance.

6.4 Risk Premia

After the American War for Independence, the Continental Congress owed approximately $40 million in foreign loans to France, Spain, and Holland and certificates of indebtedness to the American public. The Congress confronted substantially higher long term yields than the UK even though the UK then had a high debt-to-GDP ratio. This situation spawned a lively debate in the US about whether and how to service wartime debts. Ultimately, Alexander Hamilton and others persuaded Congress to repay the foreign debt at face value and issue new bonds to refinance the domestic certificates. Hamilton claimed that following through on that policy could eventually acquire for the US a reputation for servicing its debts that would reduce US interest rates to the lower levels then paid by the UK government.

We use our figure 8 yield curves to quantify whether and when Hamilton's hopes were fulfilled. The figure shows yields-to-maturity on gold denominated UK consols, yields-to-maturity on hypothetical gold denominated US consols that promise the same coupon flows as the UK consols, and the 10-year gold denominated, zero coupon yields on US treasuries.[32] We plot a yield-to-maturity on gold denominated UK consols because almost all UK government bonds were consols, so that is the only UK yield that can be reliably estimated. Notice that the long-term yield on US government debt exhibits a downward trend, falling from close to 8% at the beginning of the nineteenth century to around 2% at the end of the century. Second, notice that the US hypothetical consol was persistently higher than the UK long-term yield until the 1880s when the two series converge. In this sense, despite the Federal government's having serviced War of Independence IOUs, admittedly with substantial haircuts to domestic creditors, and having completely retired all debt by the mid 1830s, it wasn't until the late nineteenth century that Hamilton's hopes were realized. Finally, notice that the US 10-year zero does not necessarily align with the yield-to-maturity on the hypothetical US consol. The difference arises because the

[32]The UK consol yield is the series "Spliced consol yield 1753-2015, corrected for Goschen's conversion issues" from Thomas and Dimsdale (2017). The hypothetical, gold denominated US consols promise the same annuity coupon payments as those used in the UK consol yield series

yield-to-maturity is not a particular zero-coupon yield but rather the average of the zero coupon yields at different maturities and so, unlike the 10-year yield, the yield-to-maturity incorporates both the short and long ends of the yield curve.[33] Since the yield-to-maturity on the hypothetical gold denominated US consols is the natural comparison to the UK yield to maturity and calculating the hypothetical yield-to-maturity requires the full yield curve, this exercise illustates the importance of estimating the full yield curve when attempting to compare UK and US funding costs.

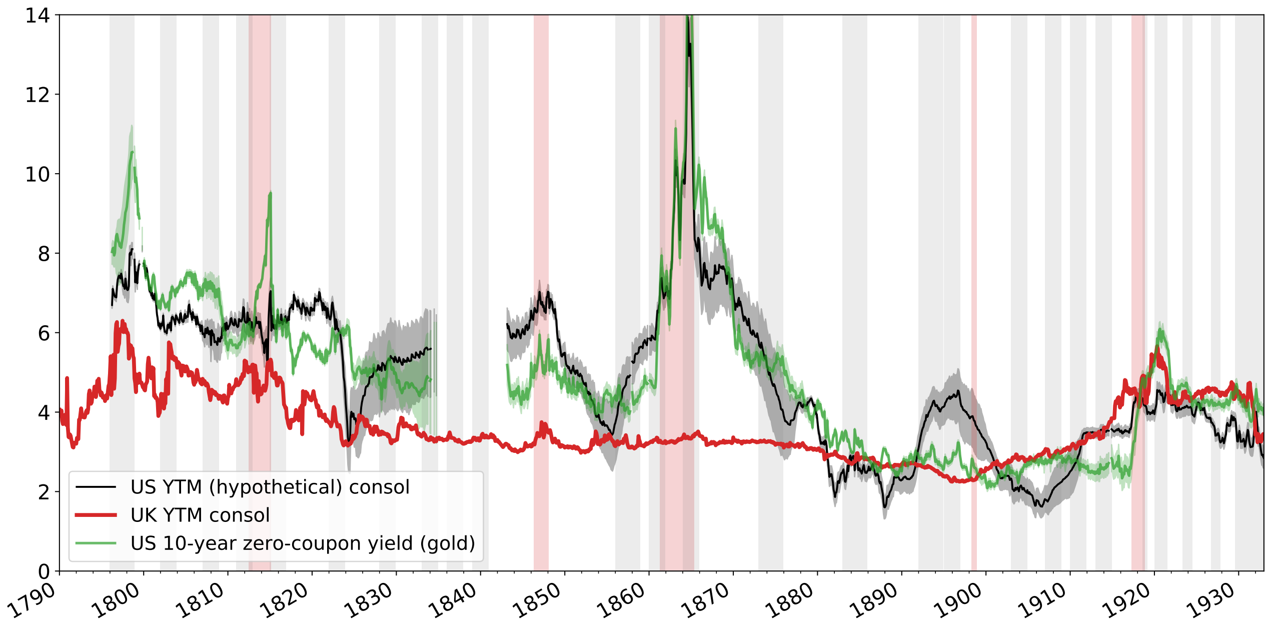

Figure 8: US and UK Long Term Yields.

The solid black line depicts the mean of our posterior estimate for the yield-to-maturity on hypothetical gold denominated US consols that promise the same coupon flows as the UK consols. The grey bands around the posterior mean depict the 95% interquantile range. The solid green line depicts the mean of our posterior estimate for the 10-year, gold denominated, zero coupon yield. The pale green bands around the posterior mean depict the 95% interquantile range. The green line depicts the UK long-term yield (implied by the 3% consol price) from Thomas and Dimsdale (2017). The light gray intervals depict recessions as dated by Davis (2006) for the 1796-1914 period and NBER recessions thereafter. The light red intervals depict wars (from left to right: the War of 1812, the Mexican-American War, the Civil War, the Spanish-American War, and World War I).

The difference between the yield-to-maturity on the UK consol and the hypothetical US consols probably reflects different haircut risks. UK debt was considered a 'safe-asset' during the nineteenth century, whereas many military and political incidents probably induced investors to regard nineteenth century US debt to be risky. We can make this claim more precise by using our pricing formulas. The yield-to-maturity on an annuity with gold coupon payments $\overline{m}$ and price p_t is the rate $\bar{y}_t$ that solves:

$$p_t = \sum_{j=1}^{\infty} \exp\left(-\bar{y}_t\right)^j \bar{m} \tag{6.1}$$

Let $\bar{q}_t := \exp\left(-\bar{y}_t\right)$. In lemma 2 in appendix B.3, we show that combining equation (6.1) with equation (4.5)

[33] We derive the connection formally in appendix B.3.

gives the following expression for the yield-to-maturity:

$$\bar{q}_t = 1 - \frac{1}{\sum_{j=0}^{\infty} q_t^{(j,g)}}$$

Let lowercase letters represent US prices and yields and let capital letters represent UK prices and yields. Then from corollary 1 in appendix B.3, we have that the difference between the US and UK consol yields is:

$$\bar{y}_t - \bar{Y}_t \approx \frac{\sum_{j=0}^{\infty}\left(q_t^{(j,g)} - Q_t^{(j,g)}\right)}{\left(\sum_{j=0}^{\infty} q_t^{(j,g)}\right)\left(\sum_{j=0}^{\infty} Q_t^{(j,g)}\right)} \tag{6.2}$$

where $\bar{y}_t$ and $q_t^{(j,g)}$ are yields-to-maturity and zero-coupon prices in the US and $\bar{Y}_t$ and $Q_t^{(j,g)}$ are yield-to-maturity and zero-coupon prices in the UK. Imposing that haircut risk is zero in the UK implies that the spread between US and UK zero-coupon bond prices is

$$q_t^{(j,g)} - Q_t^{(j,g)} = \mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]\left(\underbrace{\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]}_{\substack{\text{Expected gold}\\ \text{inflation in US}}}\underbrace{\mathbb{E}_t\left[\xi_{t+j}\right]}_{\substack{\text{US haircut}\\ \text{probability}}}\underbrace{\left(1 + \text{Cov}_t\left(\frac{\frac{S_{t+j}}{S_t}}{\mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]}, \frac{\xi_{t+j}}{\mathbb{E}_t\left[\xi_{t+j}\right]}\right)\right)}_{\text{Risk premium on US haircut risk}} - \underbrace{\mathbb{E}_t\left[\frac{E_{t+j}^{(n)}}{E_t^{(n)}}\right]}_{\substack{\text{Expected gold}\\ \text{inflation in UK}}}\right)$$

If gold inflation expectations were similar in the US and UK during the gold standard[34], then we can interpret the difference between the US and UK consol yields in figure 8 as reflecting the risk premium on US Federal debt. Under this interpretation, figure 8 suggests that US federal debt traded with a risk premium until the late nineteenth century when it became an alternative 'safe-asset' to UK consols. Evaporation of those risk-premia signals a realignment of global finance that ultimately led US government debt to replace UK debt as a global 'safe-asset' during and after the years of the Bretton Woods arrangement.

Estimating haircut risk: In principle, we could attempt to use UK yields to estimate haircut risk. However, we face the major challenge that we only observe the prices of UK consols. This means that, to make progress, we would need to impose a one-dimensional functional parametrization of $\mathbb{E}_t[\xi_{t+j}]$. Here is one way to do this. Suppose that government haircuts are governed by a two-state Markov Chain with default as an absorbing state. Let $\mathfrak{p}_t$ be bondholders' perceived probability of default in period t and assume that they use the two-state Markov Chain to forecast future cash-flows. For simplicity, suppose that upon default, government bonds pay 0. These assumptions imply $\mathbb{E}_t[\xi_{t+j}] = (1 - \mathfrak{p}_t)^j$. In addition, suppose that bondholders' are risk-neutral in the sense that $\text{cov}_t\left(\frac{S_{t+j}}{S_t},\ \xi_{t+j}^{(i)}\right) = 0$. In this special case, we have:

$$q_t^{(j,n)} - Q_t^{(j,g)} = \underbrace{\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]}_{\substack{\text{Expected gold}\\ \text{inflation in US}}}\underbrace{(1 - \mathfrak{p}_t)^j}_{\substack{\text{US haircut}\\ \text{probability}}} - \underbrace{\mathbb{E}_t\left[\frac{E_{t+j}^{(n)}}{E_t^{(n)}}\right]}_{\substack{\text{Expected gold}\\ \text{inflation in UK}}}$$

which we could combine with equation (6.2) to estimate a haircut probability $\mathfrak{p}_t$. Of course, this particular example imposes strong assumptions and ignores the possibility of varying convenience yields on US and UK federal debt. We leave the complicated task of resolving the estimation of haircut risk to future work.

[34] We have not estimated inflation expectations during the nineteenth century in the UK, but this seems like a reasonable prior given that both countries were on the gold standard.

7 Greenback Dollar Yield Curves And Exchange Rate Expectations: 1862-1878

Earlier sections have focused on the gold denominated yield curve, which we think of as our "baseline" for the gold standard period of 1790-1933. As discussed in sections 4 and 5, we also estimate a "conversion multiplier" that allows us to convert the gold denominated yield curve into a greenback denominated yield curve during this period. The estimated greenback and gold 10-year yields are shown together in figure 9. The greenback denominated yield is systematically below the gold denominated yield. This is because investors expected a return to the gold standard post Civil War and so expected greenbacks to appreciate in value. This meant that they were willing to accept a low greenback yield.[35]

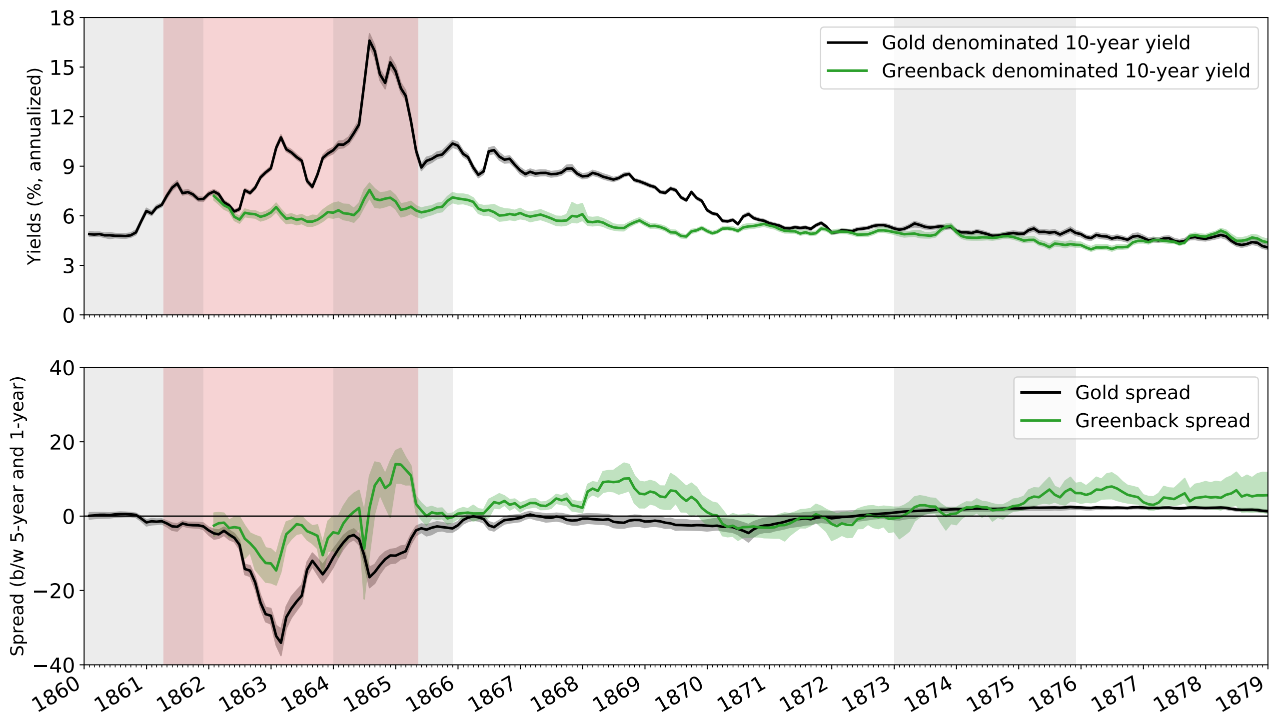

Figure 9: Yield Curves. The black line is the 10-year gold denominated zero-coupon yield curve. The green line is the 10-year greenback denominated zero-coupon yield curve. The light gray intervals depict recessions as dated by Davis (2006).

Our approach allows us to infer how investors' expectations about the greenback-dollar exchange rate evolved during and after the Civil War. Figure 10 shows our estimate for the expected Gold/Greenback exchange rate 10 years into the future at each date. As can be seen, 10 year exchange rate expectations moved very little during the Civil War. In this sense, there was a very strong "nominal anchor" throughout the Civil War.

We elaborate on this point in figure 11, which shows expected gold/green back exchange rate paths at different dates during the Civil War. On each plot, a black line shows the path of the gold/greenback exchange rate, P_t, up until a particular date, the gray line shows the continuation of the realized gold/greenback exchange rate after that date, and the orange line shows our estimates of investors' expectations about paths of the

[35] Roll (1972) makes a similar point when he discusses the greenback yield through this period.

gold/greenback exchange. Evidently, throughout the War (1861-65), investors expected a rapid return to the gold standard in the post war period. This was true even during the large drops in the value of the greenback that occurred in 1863 and 1864 in response to bad news from the war front. Thus, even in the face of very high greenback inflation during the War, expectations of a rapid resumption of greenback convertibility at par seemed to prevail. However, after the War, bond holders became less optimistic about a rapid return to gold. It is enlightening to stare at the post-war panels with a copy of (Dewey, 1922, p. 340-345) in hand and to seek explanations for this pattern there in terms of fiscal-monetary decisions made by the Congress and Treasury. Dewey describes how after mid 1866 Congress postponed measures designed to return to the gold standard. Our estimates indicate that by the mid 1870s investors thought that discrepancies between gold and greenback prices would persist indefinitely.

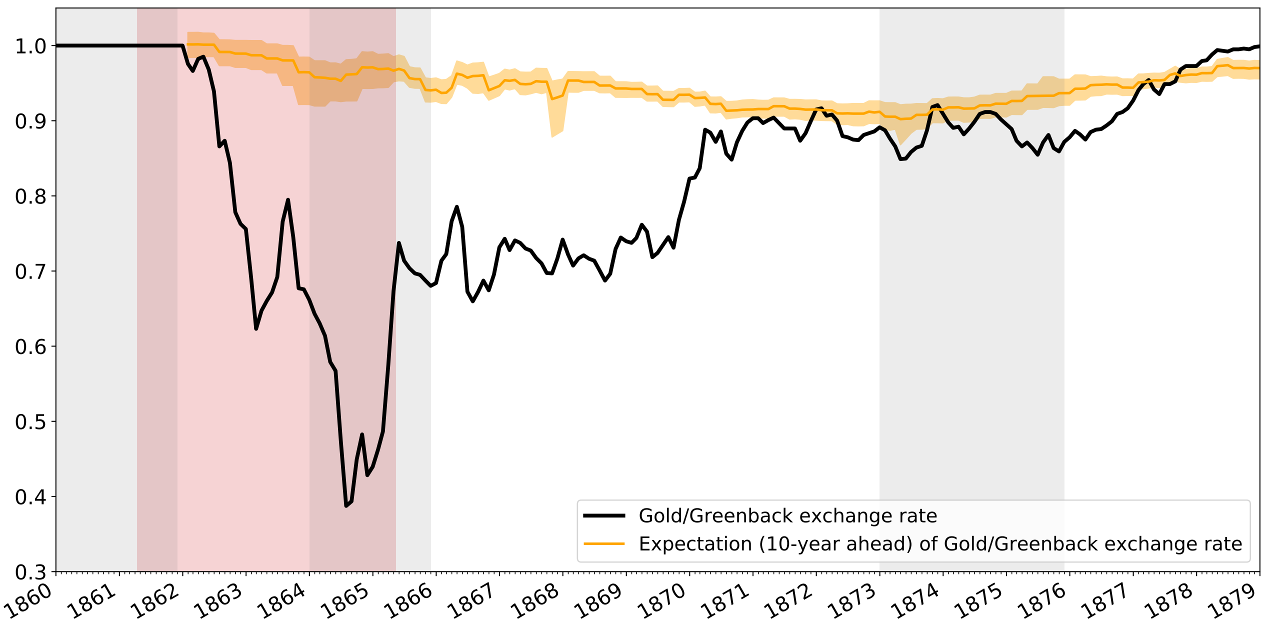

Figure 10: Long Term Exchange Rate Expectations. The black line shows the path of the gold/greenback exchange rate, P_t. The orange line shows the median of our posterior estimate for the expected Gold/Greenback exchange rate 10 years into the future at each date. The orange shaded area is the 95% interquantile range for our estimate.

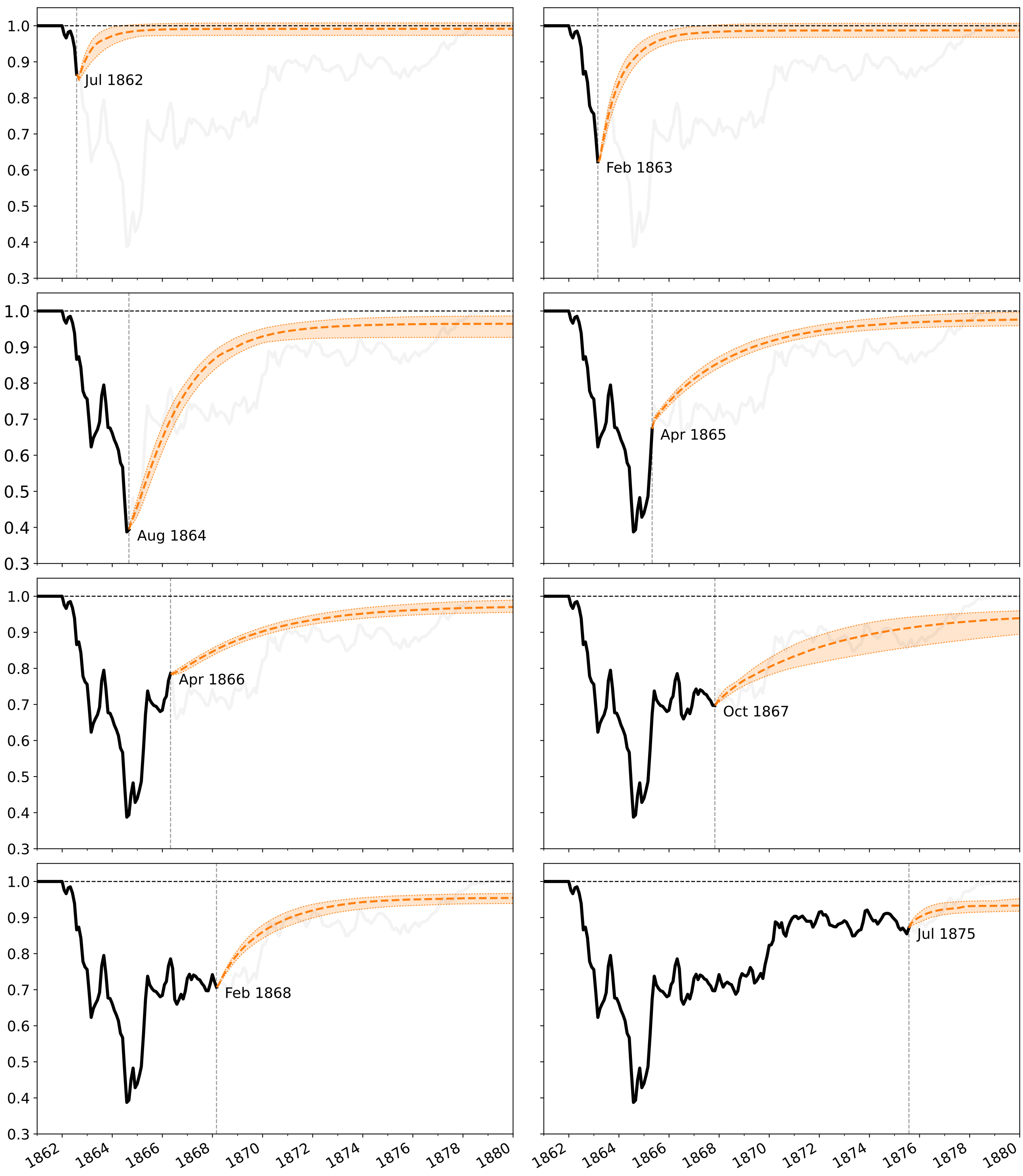

Figure 11: Expected Gold/Greenback Exchange Rate. On each plot, the black line shows the path of the gold/greenback exchange rate, P_t, up until a particular date. The gray line shows the continuation of the realized gold/greenback exchange rate after highlighted date. The dashed orange line shows our model's estimate of investors' expectations about the path of the gold/greenback exchange. The orange shaded area is the 95% interquantile range.

8 Real Yield Curves And War Finance: 1790-2020

During the late 18th and early 19th centuries, the UK serviced high debt-GDP ratios at low interest rates. US statesmen disagreed about whether the US could and should foster a similar outcome. Alexander Hamilton advocated building a reputation for repayment so that the US could on occasions run large deficits and build infrastructure. Thomas Jefferson advocated low Federal taxes and spending and a limited Federal borrowing capacity, partly because that would prevent the US from supporting a standing army and becoming entangled in foreign adventures. To quantify how this dispute played out, we need to study the *real* financing costs that the US Federal government faced when issuing bonds throughout its history. We address this by bringing together our nominal yield curve estimates for 1790-1947 (both gold and greenback denominated), combining them with existing nominal yield curve estimates from 1947-2020, and then constructing *nominal* and *real* yield curve series for 1790-2020.

To construct an *ex ante* real yield curve series, we must estimate inflation expectations at various horizons. Define currency n inflation between period t and $t+j$ as $\Pi_t^{(j,n)} := e_t^{(n)}/e_{t+j}^{(n)}$ and let $\pi_t^{(j,n)}$ denote the logarithm of $\Pi_t^{(j,n)}$. We use Assumption 4 to obtain a "risk-neutral" approximation of the *ex-ante* real yield:

$$r_t^{(j,n)} := y_t^{(j,n)} + \frac{1}{j} \log \mathbb{E}_t \left[\exp\Big(- \pi_t^{(j,n)} \Big) \right] \tag{8.1}$$

Researchers have developed sophisticated techniques for estimating inflation expectations (the second term) that incorporate macroeconomic data and theory. In principle, we could implement similar tools for the post WW1 period. However, extending these techniques back into the nineteenth century is non-trivial because there is limited reliable macroeconomic data available. For this paper, we apply our "macro-theory-lite" approach to estimating inflation expectations because we can apply it consistently throughout our entire sample. Appendix D describes a flexible model of inflation expectations—very much in the spirit of the model in Assumptions 5 and 7—that we use to estimate the second term in (8.1).

The results are shown in figure 12, which plots our estimates of 5-year zero-coupon real yields on US treasuries, our estimates of 5-year zero-coupon nominal yields on US Treasuries, US surpluses as percentages of GDP, our estimates of 5-year inflation expectations, and the rolling realized 5-year inflation. As in previous plots, gray shaded areas are recessions and red shaded areas are wars. Evidently, big deficits during the War of 1812 and the Civil War coincided with high real yields. This is in stark contrast to the US experience during the 20th century when it was able to finance large deficits during WW1, WW2, and the Depression at low real yields. The figure suggests both Hamilton and Jefferson were both prophetic. Hamilton's hopes of low interest rate deficit financing were eventually realized in the early 20th century. However, as Jefferson feared, achievement of a low financing cost regime coincided with the US introducing a big standing army and frequent participation in foreign wars.

Figure 12 sheds new light on an economic history literature that, starting with Evans (1985, 1987), has concluded that during the nineteenth century there was not a strong association between interest costs and deficits. To conclude that, previous papers used the composite series in figures 5 and 6 (the green, orange and blue lines). Our analysis indicates that that series substantially underestimates increases in yields on US Federal debt during episodes of large 19th century government deficits. One way to reconcile our analysis with this literature would be to argue that yields on US municipal and corporate bonds were not highly correlated with surpluses even though yields on US Federal bonds were. We leave a detailed analysis of nineteenth century municipal and corporate yields for future work.

Figure 12 also sheds light on the evolution of inflation expectations. For most of the nineteenth century, gold inflation expectations were anchored around zero, which kept real and gold denominated yields close

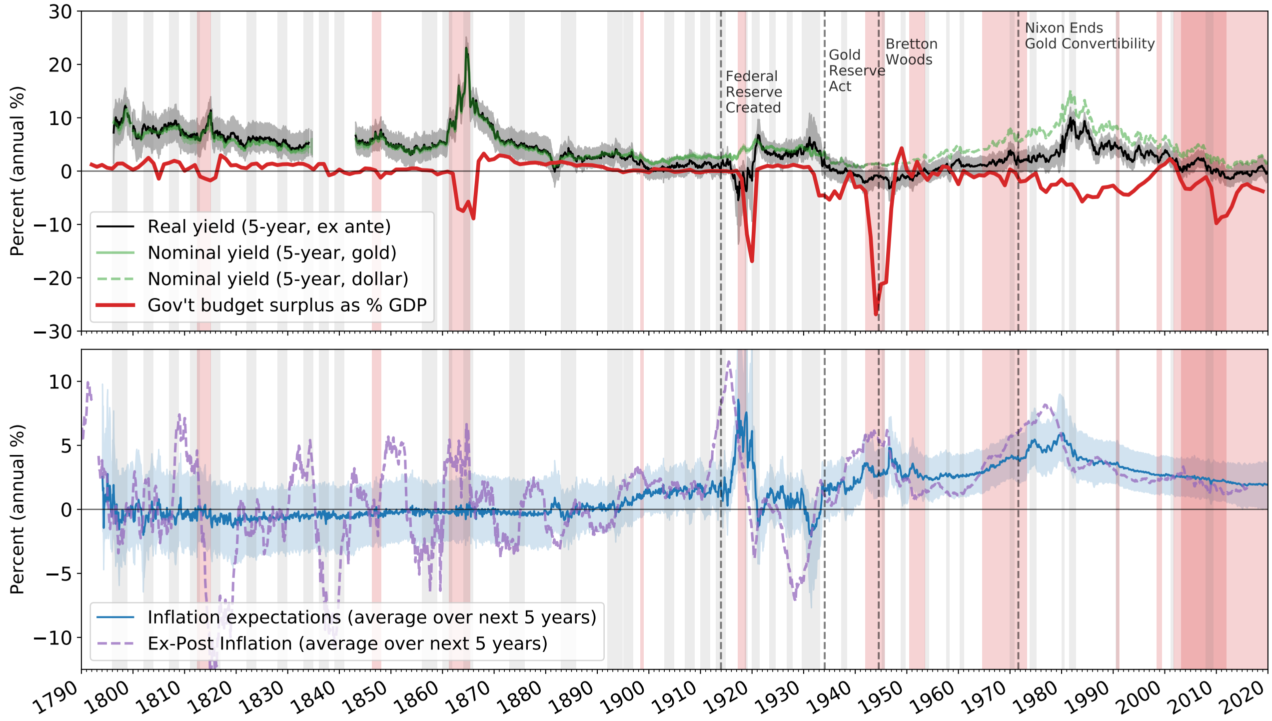

Figure 12: US Budget Surpluses, Real Bond Yields, and Inflation Expectations.

Top plot: The solid black line depicts the mean of our posterior estimate for the 5-year, gold denominated, zero coupon yield. The grey bands around the posterior mean depict the 95% interquantile range. The solid green line depicts the mean of our posterior estimate for the 5-year gold denominated yield. The dashed green line depicts the mean of our posterior estimate for the 5-year greenback denominated yield. The solid red line shows US surplus as a percentage of GDP. The light gray intervals depict recessions as dated by Davis (2006) for the 1796-1914 period and NBER recessions thereafter. The light red intervals depict major wars (from left to right: the War of 1812, the Mexican-American War, the Civil War, the Spanish-American War, World War I, and World War II). *Bottom plot:* The solid blue line depicts the mean of our posterior estimate for 5-year inflation expectations (expressed as an annualized %). The dashed purple line depicts realized inflation for the 5-year period corresponding to the inflation expectations (expressed as an annualized %). From 1790-1933, both series refer gold inflation. From 1934-2020, both series refer to greenback inflation. The pale blue bands around the posterior mean for 5-year inflation expectations depict the 95% interquantile range.

together. This was true even during the Civil War when gold coins and greenback dollars coexisted and there was significant greenback inflation. The story starts to change in the 1890s when gold inflation expectations became positive. A possible source of the change was the strong support from elements of both major political parties for returning to a bimetallic gold-silver standard at a mint price ratio of 16-1 when when the market price ratio had become much higher. Prospects of a return to bimetallism at an exchange rate that overvalued silver naturally made investors fear inflation. See Friedman (1990a), Friedman (1990b), and Velde and Weber (2000). Inflation expectations spiked to over 6% per annum during World War I but stabilized at around 1% per annum soon afterwards. That pattern may reflect that the US was one of the few Western countries to not formally abandon the gold standard during the war. The next major change came in 1933 when President Roosevelt signed the Gold Reserve Act that, at least for US citizens, effectively took the US off the gold standard. Inflation expectations immediately increase by approximately 4 percentage points and then remain positive through throughout the rest of the 20th century, before ultimately settling down to around 2% per annum in recent years. It is worthwhile comparing our estimates here with what Goodfriend and King (2005) describe as Paul Volcker's incredible disinflation.

To highlight and compare how different US wars were financed, we overlay time series for the Civil War

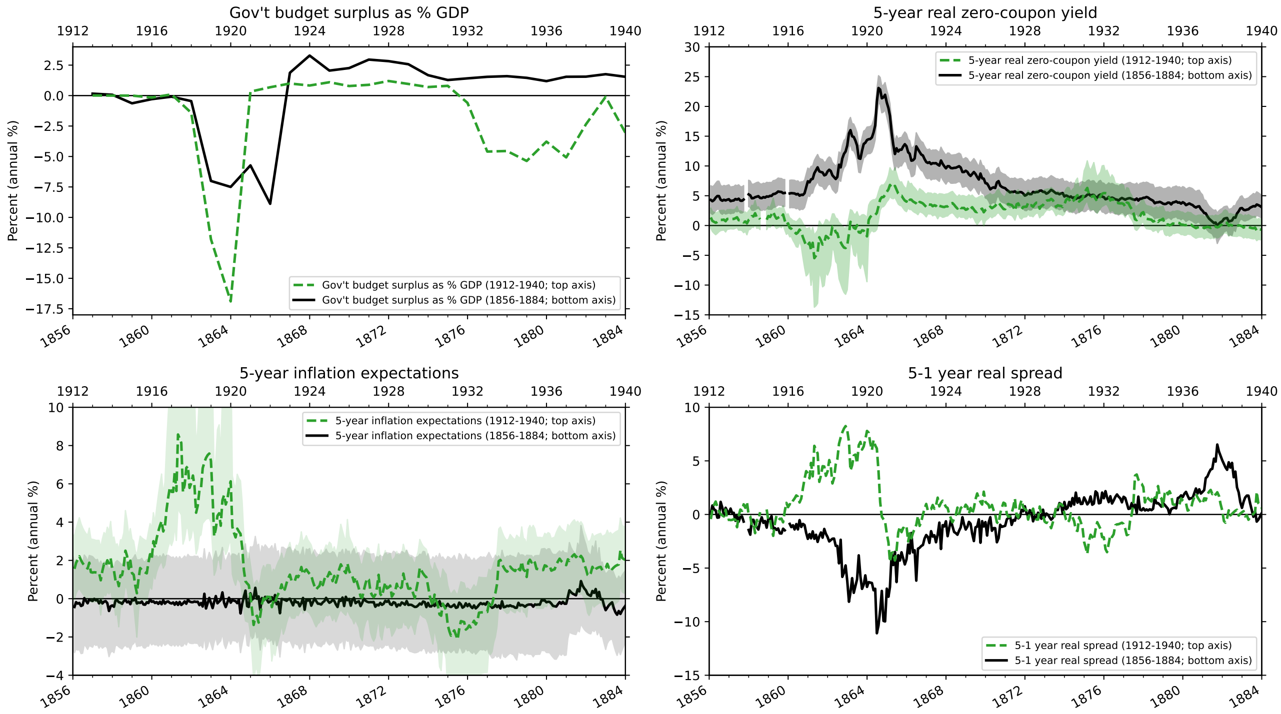

Figure 13: Financing the Civil War and World War I.

Top plot: The solid black lines and gray bands refer to time series from 1860-1884. The dashed green lines and light green bands refer to time series from 1916-1933.

and World War I in figure 13. The black lines depict surpluses, yields, inflation expectations, and term spreads for the Civil War while green lines depict the same variables for World War I. The bottom axes refer to Civil War variables while the top axes refer to World War I variables. The first two subplots emphasize that, even though the US ran a larger deficit during World War I, the 5-year real yield remained negative until around 1920 rather than increasing dramatically like it did during the Civil War. However, note that the real yield steadily increased after World War I and reverted to pre-war levels more slowly than following the Civil War. An interpretation is that monetary arrangements during the 1920s delayed the fall in yields until the 1930s Depression. The bottom right subplot shows that the spread between the real 5-year yield and the real 1-year yield moved in opposite directions during the two wars. During the Civil War, the real yield curve inverted making longer term financing relatively cheap while during World War I the spread increased and longer term financing became relatively expensive.

9 Concluding Remarks

Our research here is partly a "proof of concept": we have used Bayesian Monte Carlo methods to approximate posterior probabilities of a profligately parameterized but theoretically highly restricted statistical model. The model blends a tailored asset price asset pricing theory that bundles stochastic discount factors with haircut and exchange rate risks together with a statistical model of how quickly yield curves move over time. We have used the model's pricing errors to diagnose measurement errors and conceptual problems involving units of accounts. Here we have shown only the tip of an iceberg. Offline, we have used the model to compose "bond biographies" of some classic bonds beloved of US financial historians, for example, the Stocks of the 1790s, the Civil War-era 5-20s, and World War I Liberty and Victory Loans.

The quality and plausibility of our approximate yield curves convince us that our approach could be used in other settings where governments have issued bonds in different currencies. One example might be the 1997 Asian financial crisis where many South East Asian governments and banks issued bonds both in local currencies and in US dollars. Our estimates also might qualify as plausible inputs to subsequent research that, in the spirit of "factor models" of stochastic discount factors, would use related macro series to refine understandings of forces that drive yield curves and forecasts of important macro time series.

References

Andreasen, M. M., Christensen, J. H., and Rudebusch, G. D. (2019). Term structure analysis with big data: One-step estimation using bond prices. *Journal of Econometrics*, 212:26–46.

Ang, A., Piazzesi, M., and Wei, M. (2006). What does the yield curve tell us about gdp growth? *Journal of econometrics*, 131(1-2):359–403.

Annaert, J., Claes, A. G., Ceuster, M. J. D., and Zhang, H. (2013). Estimating the spot rate curve using the nelson–siegel model: A ridge regression approach. *International Review of Economics and Finance*, 27:482–496.

Barnard, J., McCulloch, R., and Meng, X.-L. (2000). Modeling covariance matrices in terms of standard deviations and correlations, with application to shrinkage. *Statistica Sinica*, 10(4):1281–1311.

Bayley, R. A. (1882). *The National Loans of the United States, from July 4, 1776, to June 30, 1880*. Number 246. US Government Printing Office.

Betancourt, M. (2018). A conceptual introduction to hamiltonian monte carlo.

Bezanson, J., Edelman, A., Karpinski, S., and Shah, V. B. (2017). Julia: A fresh approach to numerical computing. *SIAM review*, 59(1):65–98.

Björk, T. and Christensen, B. J. (1999). Interest rate dynamics and consistent forward rate curves. *Mathematical Finance*, 9(4):323–348.

Board of Governors of the Federal Reserve System (1943). *Banking and Monetary Statistics, 1914-1941*. Board of Governors of the Federal Reserve System, Washington D.C.

Cecchetti, S. G. (1988). The case of the negative nominal interest rates: New estimates of the term structure of interest rates during the great depression. *Journal of Political Economy*, 96(6):1111–1141.

Chernow, R. (2001). *The House of Morgan: An American Banking Dynasty and the Rise of Modern Finance*. Grove Press.

Christensen, J. H., Diebold, F. X., and Rudebusch, G. D. (2011). The affine arbitrage-free class of nelson–siegel term structure models. *Journal of Econometrics*, 164(1):4–20.

Cogley, T. (2005). Changing beliefs and the term structure of interest rates: Cross-equation restrictions with drifting parameters. *Review of Economic Dynamics*, 8(2):420–451. Monetary Policy and Learning.

Cogley, T. and Sargent, T. J. (2005). Drifts and volatilities: monetary policies and outcomes in the post wwii us. *Review of Economic Dynamics*, 8(2):262–302. Monetary Policy and Learning.

Dahlquist, M. and Svensson, L. E. O. (1996). *Scandinavian Journal of Economics*, pages 163–183.

Davis, J. H. (2004). An annual index of us industrial production, 1790–1915. *The Quarterly Journal of Economics*, 119(4):1177–1215.

Davis, J. H. (2006). An improved annual chronology of u.s. business cycles since the 1790s. *The Journal of Economic History*, 66(1):103–121.

Dewey, D. R. (1922). *Financial History of the United States, Eighth edition.* Longmans, Green, and Co., New York.

Diebold, F. X. and Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130(2):337–364.

Diebold, F. X., Li, C., and Yue, V. Z. (2008). Global yield curve dynamics and interactions: A dynamic Nelson-Siegel approach. *Journal of Econometrics*, 146(2):351–363.

Diebold, F. X., Piazzesi, M., and Rudebusch, G. D. (2005). Modeling bond yields in finance and macroeconomics. *American Economic Review*, 95(2):415–420.

Edwards, S. (2018). *American Default: The Untold Story of FDR, the Supreme Court, and the Battle over Gold*. Princeton University Press, Princeton, New Jersey.

English, W. B. (1996). Understanding the costs of sovereign default: American state debts in the 1840's. *The American Economic Review*, 86(1):259–275.

Evans, P. (1985). Do large deficits produce high interest rates? *The American Economic Review*, 75(1):68–87.

Evans, P. (1987). Interest rates and expected future budget deficits in the united states. *Journal of political Economy*, 95(1):34–58.

Filipović, D. (1999). A note on the nelson–siegel family. *Mathematical finance*, 9(4):349–359.

Friedman, M. (1990a). Bimetallism Revisited. *Journal of Economic Perspectives*, 4(4):85–104.

Friedman, M. (1990b). The Crime of 1873. *Journal of Political Economy*, 98(6):1159–1194.

Goodfriend, M. and King, R. G. (2005). The incredible Volcker disinflation. *Journal of Monetary Economics*, 52(5):981–1015.

Gürkaynak, R. S., Sack, B., and Wright, J. H. (2007). The us treasury yield curve: 1961 to the present. *Journal of monetary Economics*, 54(8):2291–2304.

Hall, G. J., Payne, J., Sargent, T. J., and Szőke, B. (2018). US federal debt 1776-1940: Prices and quantities. https://github.com/jepayne/US-Federal-Debt-Public.

Hall, G. J. and Sargent, T. J. (2011). Interest Rate Risk and Other Determinants of Post-WWII US Government Debt/GDP Dynamics. *American Economic Journal: Macroeconomics*, 3(3):192–214.

Hamilton, J. D., Harris, E. S., Hatzius, J., and West, K. D. (2016). The Equilibrium Real Funds Rate: Past, Present, and Future. *IMF Economic Review*, 64(4):660–707.

Hansen, L. P. and Jagannathan, R. (1997). Assessing specification errors in stochastic discount factor models. *The Journal of Finance*, 52(2):557–590.

Hoffman, M. D. and Gelman, A. (2014). The no-u-turn sampler: Adaptively setting path lengths in hamiltonian monte carlo. *Journal of Machine Learning Research*, 15(47):1593–1623.

Homer, S. and Sylla, R. E. (2004). *A history of interest rates.* Rutgers University Press.

Jordà, Ò., Knoll, K., Kuvshinov, D., Schularick, M., and Taylor, A. M. (2019). The rate of return on everything, 1870–2015. *The Quarterly Journal of Economics*, 134(3):1225–1298.

Kreps, D. M. (1998). *Anticipated Utility and Dynamic Choice (1997)*, page 242–274. Econometric Society Monographs. Cambridge University Press.

Krippner, L. (2015). A theoretical foundation for the nelson–siegel class of yield curve models. *Journal of Applied Econometrics*, 30(1):97–118.

Lewandowski, D., Kurowicka, D., and Joe, H. (2009). Generating random correlation matrices based on vines and extended onion method. *Journal of Multivariate Analysis*, 100(9):1989 – 2001.

Martin, J. G. (1886). *Martin's Boston Stock Market: Eighty-eight Years, from January 1, 1798, to January, 1886*. The author.

McGrane, R. C. (1935). *Foreign Bondholders and American State Debts*. New York: The Macmillan Company.

Mitchell, W. C. (1903). *A History of the Greenbacks with Special Reference to the Economic Consequences of Their Issue: 1862-1865*. University of Chicago Press, Chicago, Illinois.

Mitchell, W. C. (1908). *Gold, Prices, and Wages under the Greenback Standard*. University of California, Berkeley, California.

Nelson, C. R. and Siegel, A. F. (1987). Parsimonious modeling of yield curves. *The Journal of Business*, 60(4):473–489.

Noyes, A. D. (1909). *Forty Years of American Finance: A Short Financial History of the Government and People of the United States Since the Civil War, 1865-1907*. GP Putnam's Sons.

Officer, L. H. and Williamson, S. H. (2021). Explaining the measures of worth. *Measuring Worth*.

Papp, T. K., Aluthge, D., JackRab, Widmann, D., TagBot, J., and Piibeleht, M. (2021). tpapp/dynamichmc.jl: v3.1.0. Julia package version 3.1.0.

Piazzesi, M. and Schneider, M. (2007). Equilibrium yield curves. *NBER Macroeconomics Annual*, 21:389–472.

Razaghian, R. (2002). Financial credibility in the united states: The impact of institutions, 1789-1860. Data made available through the International Center for Finance at Yale University.

Register's Office, U. T. (1886). *Statement of Receipts and Expenditures of the Government (by Warrants) from July 1, 1855, to June 30, 1885 and Statement of Principal of Public Debt from 1791 to 1836 by Issues and Redemptions, and From January 1, 1836, to June 30, 1885 by Warrants*. Government Printing Office, Washington DC.

Roll, R. (1972). Interest rates and price expectations during the civil war. *Journal of Economic History*, 32:476–498.

Rothbard, M. N. (2002). *A History of Money and Banking in the United States: the Colonial Era to World War II*. Ludwig von Mises Institute, Auburn, Alabama.

Shiller, R. J. (2015). *Irrational Exuberance, 3rd edition*. Princeton University Press, revised and expanded third edition edition.

Siegel, J. J. (1992). The real rate of interest from 1800-1990. *Journal of Monetary Economics*, 29(2):227–252.

Stock, J. H. and Watson, M. (2003). Forecasting output and inflation: The role of asset prices. *Journal of Economic Literature*, 41(3):788–829.

Svensson, L. E. (1994). Estimating and interpreting forward interest rates: Sweden 1992 - 1994. Working Paper 4871, National Bureau of Economic Research.

Svensson, L. E. O. (1995). Estimating forward interest rates with the extended nelson and siegel method. *Sveriges Riksbank Quarterly Review*, 3.

Sylla, R. E., Wilson, J., and Wright, R. E. (2006). Early u.s. security prices. *http://eh.net/databases/early-us-securities-prices.*

Szőke, B. (2021). Estimating robustness. *Journal of Economic Theory*, pages 105–225.

Thomas, R. and Dimsdale, N. (2017). A millennium of uk data. *Bank of England OBRA dataset.*

United States Department of the Treasury (1869-2015). *Monthly Statement of the Public Debt.* U.S. Treasury Department.

Velde, F. R. and Weber, W. E. (2000). A Model of Bimetallism. *Journal of Political Economy*, 108(6):1210–1234.

Warren, G. F., Pearson, F. A., Stoker, H. M., et al. (1932). Wholesale prices for 213 years, 1720 to 1932.

A Additional Data Description

A.1 Construction of Recession bands

For the 1796-1914 period we use recession dates from Davis (2006). These are derived solely from the Davis (2004) annual industrial production index. The Davis index incorporates 43 annual series in the manufacturing and mining industries in a manner similar to the Federal Reserve Board's present-day industrial production index. For this reason, we regard it as an improvement over earlier more qualitative approaches of dating pre-World War I business cycles. Since the data used to date peaks and troughs is annual, the methodology is quite simple: A year immediately preceding an absolute decline in the aggregate level of Davis's industrial production index defines a peak, and the last consecutive decline following a peak defines a trough (Davis, 2006). For the 1915-present period we use recession dates from the NBER.

B Additional Theory

B.1 Supplementary Proofs on Currency Risk Premia

Proof of Lemma 1. The dollar n price can expressed as[36]:

$$
\begin{aligned}
q_t^{(j,n)} &= \mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right)\xi_{t+j}\right] \\
&\approx \mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]\mathbb{E}_t\left[\xi_{t+j}\right] + \mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]\mathrm{Cov}_t\left(\frac{S_{t+j}}{S_t}, \xi_{t+j}\right) \\
&\quad + \mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]\mathrm{Cov}_t\left(\xi_{t+j}, \frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right) + \mathbb{E}_t\left[\xi_{t+j}\right]\mathrm{Cov}_t\left(\frac{S_{t+j}}{S_t}, \frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right) \\
&= \mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]\mathbb{E}_t\left[\xi_{t+j}\right]\left(1 + \frac{\mathrm{Cov}_t\left(\frac{S_{t+j}}{S_t}, \frac{\xi_{t+j}}{\mathbb{E}_t[\xi_{t+j}]}\right)}{\mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]} + \frac{\mathrm{Cov}_t\left(\xi_{t+j}, \frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right)}{\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]\mathbb{E}_t\left[\xi_{t+j}\right]} + \frac{\mathrm{Cov}_t\left(\frac{S_{t+j}}{S_t}, \frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right)}{\mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]}\right)
\end{aligned}
$$

---

[36]Under the assumption that $\mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t} - \mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]\right)\left(\frac{e_{t+j}^{(n)}}{e_t^{(n)}} - \mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]\right)\left(\xi_{t+j} - \mathbb{E}_t\left[\xi_{t+j}\right]\right)\right] \approx 0$. In continuous time, the approximation sign would be replaced by an equals sign.

and so the difference between a dollar n yield and the risk free real yield is approximately[37]:

$$
\begin{aligned}
y_t^{(j,n)} - \hat{y}_t^{(j)} = & -\frac{1}{j}\log\left(\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]\right) - \frac{1}{j}\log\left(\mathbb{E}_t\left[\xi_{t+j}\right]\right) \\
& -\frac{1}{j}\log\left(1 + \frac{\mathrm{Cov}_t\left(\xi_{t+j}, \frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right)}{\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]\mathbb{E}_t\left[\xi_{t+j}\right]} + \frac{\mathrm{Cov}_t\left(\frac{S_{t+j}}{S_t}, \frac{\xi_{t+j}}{\mathbb{E}_t[\xi_{t+j}]}\right)}{\mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]} + \frac{\mathrm{Cov}_t\left(\frac{S_{t+j}}{S_t}, \frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right)}{\mathbb{E}_t\left[\frac{S_{t+j}}{S_t}\right]\mathbb{E}_t\left[\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right]}\right) \\
\approx & \underbrace{-\frac{1}{j}\log\left(\frac{\mathbb{E}_t\left[e_{t+j}^{(n)}\right]}{e_t^{(n)}}\right)}_{\substack{\text{Expected dollar } n \\ \text{inflation}}} + \underbrace{\frac{-1}{j}\log\mathbb{E}_t\left[\xi_{t+j}\right]}_{\text{haircut probability}} + \underbrace{-\frac{1}{j}\left(\mathrm{Cov}_t\left(\frac{\xi_{t+j}}{\mathbb{E}_t\left[\xi_{t+j}\right]}, \frac{e_{t+j}^{(n)}/e_t^{(n)}}{\mathbb{E}_t[e_{t+j}^{(n)}/e_t^{(n)}]}\right)\right)}_{\substack{\text{Risk from haircut} \\ \text{\& inflation comovement}}} \\
& + \underbrace{-\frac{1}{j}\left(\mathrm{Cov}_t\left(\frac{S_{t+j}/S_t}{\mathbb{E}_t\left[S_{t+j}/S_t\right]}, \frac{\xi_{t+j}}{\mathbb{E}_t\left[\xi_{t+j}\right]}\right)\right)}_{\substack{\text{Risk premium on} \\ \text{haircut risk}}} + \underbrace{-\frac{1}{j}\left(\mathrm{Cov}_t\left(\frac{S_{t+j}/S_t}{\mathbb{E}_t\left[S_{t+j}/S_t\right]}, \frac{e_{t+j}^{(n)}/e_t^{(n)}}{\mathbb{E}_t\left[e_{t+j}^{(n)}/e_t^{(n)}\right]}\right)\right)}_{\substack{\text{Risk premium on} \\ \text{dollar } n \text{ inflation}}}
\end{aligned}
$$

□

B.2 Zero-Coupon Bonds Having Ambiguous Denominations

In section 4, we only considered bonds whose payoff currency denominations are certain and whose only risk involves possible haircuts. We now consider a zero-coupon bond that promises to pay $\overline{m}_t^{(i,a)}$ dollars of ambiguous denomination (denoted by index a). For these bonds, pricing formulas are more complicated. Let $\gamma_{t+j}^{(i)}$ denote the probability that $m_{t+j}^{(i,a)}$ is actually made in gold dollars at time $t+j$ and that $1-\gamma_{t+j}^{(i)}$ is the probability that payment is made in greenback dollars. The price of gold denominated bonds with ambiguous payment currency is:

$$
\begin{aligned}
\epsilon_t^{(g)} p_t^{(i,j,a)} &= \mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\gamma_{t+j}^{(i)} e_{t+j}^{(g)} m_{t+j}^{(i,a)} + (1-\gamma_{t+j}^{(i)}) e_{t+j}^{(d)} m_{t+j}^{(i,a)}\right)\right] \\
&= \mathbb{E}_t\Big[\left(\frac{S_{t+j}}{S_t}\right)\left(\gamma_{t+j}^{(i)} e_{t+j}^{(g)} + (1-\gamma_{t+j}^{(i)}) e_{t+j}^{(d)}\right)\xi_{t+j}\Big]\overline{m}_{t+j}^{(i,a)}.
\end{aligned}
$$

where, as before, $\xi_{t+j} = m_{t+j}^{(i,a)}/\overline{m}_{t+j}^{(i,a)}$. To create formulas comparable to those in the previous sections, we can rewrite this equation as:

$$
\begin{aligned}
p_t^{(i,j,a)} &= \mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\gamma_{t+j}^{(i)}\frac{e_{t+j}^{(g)}}{e_t^{(g)}} + (1-\gamma_{t+j}^{(i)})\frac{e_{t+j}^{(d)}}{e_t^{(g)}}\right)\xi_{t+j}\right]\overline{m}_{t+j}^{(i,a)} \\
&= \mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\frac{e_{t+j}^{(g)}}{e_t^{(g)}}\right)\left(\gamma_{t+j}^{(i)} + (1-\gamma_{t+j}^{(i)})P_{t+j}\right)\xi_{t+j}\right]\overline{m}_{t+j}^{(i,a)}
\end{aligned}
$$

[37]Using the approximation that $\log(1+x) \approx x$

and define the price:

$$q_t^{(i,j,a)} := \mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\underbrace{\left(\frac{e_{t+j}^{(g)}}{e_t^{(g)}}\right)}_{\substack{\text{Gold Inflation}\\ \text{Risk}}}\underbrace{\left(\gamma_{t+j}^{(i)}+(1-\gamma_{t+j}^{(i)})P_{t+j}\right)}_{\substack{\text{Denomination}\\ \text{Risk}}}\underbrace{\xi_{t+j}}_{\substack{\text{Haircut}\\ \text{Risk}}}\right]$$

so that the price of a bond with ambiguous denomination is

$$p_t^{(j,n)} := q_t^{(i,j,a)}\overline{m}_{t+j}^{(i,a)}.$$

Assumption 9. Conditional on P_{t+j}, denomination risk, $\gamma_{t+j}^{(i)}$, is independent of all other variables. $\gamma_{t+j}^{(i)}$ is also independent of i and so we can write $\gamma_{t+j}^{(i)} = \gamma_{t+j}$, for all ambiguously denominated bonds i.

Assumption 9 allows us to represent $q_t^{(j,a)}$ as:

$$\begin{aligned}
q_t^{(j,a)} &= \mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\frac{e_{t+j}^{(g)}}{e_t^{(g)}}\right)\xi_{t+j}\left(\gamma_{t+j}+(1-\gamma_{t+j})P_{t+j}\right)\right]\\
&= \mathbb{E}_t\left[\mathbb{E}_t\left[\gamma_{t+j}\middle|P_{t+j}\right]\mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\frac{e_{t+j}^{(g)}}{e_t^{(g)}}\right)\xi_{t+j}\middle|P_{t+j}\right]\right]\\
&\quad+\mathbb{E}_t\left[\left(1-\mathbb{E}_t\left[\gamma_{t+j}\middle|P_{t+j}\right]\right)\mathbb{E}_t\left[\left(\frac{S_{t+j}}{S_t}\right)\left(\frac{e_{t+j}^{(g)}}{e_t^{(g)}}\right)\xi_{t+j}P_{t+j}\middle|P_{t+j}\right]\right]
\end{aligned}$$

We keep our model parsimonious by parameterizing γ_{t+j} as a function of P_{t+j}. To attain comparability with earlier notation, we define the conversion multiple $w_t^{(j)}$ by:

$$w_t^{(j)} := \frac{q_t^{(j,a)}}{q_t^{(j,g)}}$$

so that we can write the price of ambiguously denominated bonds as:

$$p_t^{(i,j,a)} = q_t^{(j,g)}w_t^{(j)}\overline{m}_{t+j}^{(i,a)}$$

Remark: Risk Decomposition: The previous section assumed that we can decompose default risk on a bond denominated in currency n into orthogonal haircut and currency n inflation risks:

$$\text{Default risk} = \underbrace{\xi_{t+j}}_{\substack{\text{Haircut}\\ \text{risk}}}\underbrace{\left(\frac{e_{t+j}^{(n)}}{e_t^{(n)}}\right)}_{\substack{\text{Currency } n\\ \text{inflation risk}}}\quad s.t.\quad \xi_{t+j} \perp\!\!\!\perp \frac{e_{t+j}^{(n)}}{e_t^{(n)}}$$

A natural concern with this approach is that the government could have imposed haircuts by paying greenback dollars for what it had promised would be gold dollars. That would make haircut risk and currency inflation n risk be correlated. Introducing γ is one way of relaxing assumption 4 because with γ in play: (i) default risk can be decomposed into haircut, denomination, and price level risks, and (ii) dependence between denomination and price level risks is allowed. This means that default risk is now:

$$\text{Default risk} = \underbrace{\xi_{t+j}}_{\substack{\text{Haircut}\\ \text{Risk}}} \underbrace{(\gamma_{t+j} + (1-\gamma_{t+j})P_{t+j})}_{\substack{\text{Denomination}\\ \text{Risk}}} \underbrace{\left(\frac{e^{(n)}_{t+j}}{e^{(n)}_t}\right)}_{\substack{\text{Currency } n\\ \text{Inflation Risk}}} \quad s.t \quad \xi_{t+j} \perp\!\!\!\perp \left\{\frac{e^{(n)}_{t+j}}{e^{(n)}_t}, \gamma_{t+j}, P_{t+j}\right\}$$

Here we have partitioned haircut risk into a denomination risk component that can be correlated with currency prices and a second component that is independent of currency prices. Currency n inflation risk is the term $e^{(n)}_{t+j}/e^{(n)}_t$, which is the factor by which the goods price of currency n changes.

B.3 Connection Between Yields on Finite-Horizon Zero-Coupon Bonds and Yield-To-Maturity

Some analysts have expressed historical long-term interest rates as yields-to-maturity rather than zero-coupon yields. In this appendix, we discuss the connection between the different types of yields. A yield-to-maturity (a.k.a. an internal rate of return) is defined as a fixed discount rate, $\bar{y}^{(i,n)}$, that equates the currency n bond price to the present discounted value of its promised currency n payments. Thus, the dollar n yield-to-maturity on bond i with payments in currency n and maturity $J^{(i)}$ is the rate $\bar{y}^{(i,n)}_t$ that solves:

$$p^{(i,n)}_t = \sum_{j=1}^{J^{(i)}} \exp\left(-\bar{y}^{(i,n)}_t\right)^j \bar{m}^{(i,n)}_{t+j}$$

To compare to the zero-coupon prices, let $\bar{q}^{(i,n)}_t := \exp\left(-\bar{y}^{(i,n)}_t\right)$. The bond price can be expressed in terms of $\bar{q}^{(i,n)}_t$ as:

$$p^{(i,n)}_t = \sum_{j=1}^{J^{(i)}} \left(\bar{q}^{(i,n)}_t\right)^j \bar{m}^{(i,n)}_{t+j}. \tag{B.1}$$

Lemma 2. *Consider a bond with $J^i = \infty$ and $\bar{m}^{(i,n)}_{t+j} = \bar{m}^{(n)}$ (i.e. a fixed coupon annuity in currency n). Denote the yield-to-maturity on such a bond by $\bar{y}^{(n)}_t$ and the associated price by $\bar{q}^{(n)}_t := \exp(-\bar{y}^{(n)}_t)$. Then $\bar{q}^{(n)}_t$ can be expressed in terms of zero-coupon yields as:*

$$\bar{q}^{(n)}_t = 1 - \frac{1}{\sum_{j=0}^{\infty} q^{(j,n)}_t} \tag{B.2}$$

Proof. From equation (B.1), we have that the price of the fixed coupon annuity is:

$$\bar{p}^{(n)}_t = \sum_{j=1}^{\infty} \left(\bar{q}^{(n)}_t\right)^j \bar{m}^{(n)} = \bar{m}^{(n)} \sum_{j=1}^{\infty} \left(\bar{q}^{(n)}_t\right)^j = \bar{m}^{(n)} \left(\frac{1}{1-\bar{q}^{(n)}_t} - 1\right)$$

From equation (4.5) we also have the expression:

$$\bar{p}^{(n)}_t = \sum_{j=1}^{\infty} \bar{q}^{(j,n)}_t \bar{m}^{(n)} = \bar{m}^{(n)} \left(\sum_{j=0}^{\infty} \bar{q}^{(j,n)}_t - 1\right)$$

where $\bar{q}_t^{(0,n)} = 1$. Equating the expressions gives that:

$$\frac{1}{1 - \bar{q}_t^{(n)}} = \sum_{j=0}^{\infty} \bar{q}_t^{(j,n)}$$

and rearranging gives the desired result. $\square$

Corollary 1. *Let lowercase letters represent US prices and yields and let capital letters represent UK prices and yields. Then the difference between the US and UK consol yields is*

$$\bar{y}_t - \bar{Y}_t \approx \frac{\sum_{j=0}^{\infty}\left(q_t^{(j,g)} - Q_t^{(j,g)}\right)}{\left(\sum_{j=0}^{\infty} q_t^{(j,g)}\right)\left(\sum_{j=0}^{\infty} Q_t^{(j,g)}\right)}$$

Proof. Using equation (B.2), we have that:

$$\bar{y}_t = \log(\bar{q}_t^{(n)}) = \log\left(1 - \frac{1}{\sum_{j=0}^{\infty} q_t^{(j,n)}}\right) \approx -\frac{1}{\sum_{j=0}^{\infty} q_t^{(j,n)}}$$

and so:

$$\begin{aligned}\bar{y}_t - \bar{Y}_t &\approx -\frac{1}{\sum_{j=0}^{\infty} q_t^{(j,n)}} + \frac{1}{\sum_{j=0}^{\infty} Q_t^{(j,n)}} \\ &= \frac{\sum_{j=0}^{\infty}\left(q_t^{(j,g)} - Q_t^{(j,g)}\right)}{\left(\sum_{j=0}^{\infty} q_t^{(j,g)}\right)\left(\sum_{j=0}^{\infty} Q_t^{(j,g)}\right)}\end{aligned}$$

$\square$

Equation (B.1) indicates that the yield-to-maturity on a coupon-bearing bond is some kind of *weighted average* of zero-coupon yields, with cash-flow payments serving as weights. For the case of an annuity, the average is unweighted and reduces to equation (B.2). Because a principal payment is typically substantially larger than the coupon payments, the maturity-related zero-coupon yield gets the largest weight in the average. As a result, a yield-to-maturity on a J-maturity bond can approximate a J-period zero-coupon yield, although the quality of approximation depends on details of a bond's promised payment stream. The only exact equality is that a yield-to-maturity on a j-period zero-coupon bond coincides with the j-period zero-coupon yield, $y_t^{(j,n)}$.

The Congress and the Treasury often aimed to set coupon rates on new bonds so that initially they would sell at par. That outcome would make their yields-to-maturities equal their coupon rates. As we will see below, changes in market conditions frustrated this objective during important episodes in US history. Thus, at times of financial distress during the War of 1812 and the Civil War, Treasury debt sold at deep discounts; and during disagreements between the President and the Congress, like those in the 1890s, the Treasury issued bonds with coupon rates exceeding current yields, so that bonds sold at a premium.[38]

In remarks at a 2010 Minneapolis Fed conference, Professor V.V. Chari offered an "accounting tail wags the

[38] In 1895, after a run drained 40% of the Treasury's Gold Reserve Fund, President Grover Cleveland sought to issue debt to purchase the gold needed to replenish these reserves. But proponents of bimetallism in Congress blocked new borrowing. Accepting advice from J.P. Morgan's lawyers, the Cleveland Administration bypassed Congress and used some Civil War-era legislation to issue 30-year bonds bearing 4 percent coupons, at a time when the 10-year zero-coupon yield was below 3 percent. Controversy surrounding the issuance of these bonds helped inspire William Jennings Bryan's "Cross of Gold" Speech at the 1896 Democratic Convention. See Chernow (2001, ch5) for details.

dog" explanation of why Congresses often wanted only to market new bonds that would sell "at par".[39] Chari's explanation was that Congresses viewed themselves as stuck with Alexander Hamilton's peculiar accounting rules that told them to measure total government debt by simply adding up *undiscounted* par values of all outstanding debts, ignoring coupon values. That accounting system could provide good approximations to the value of debt only if bonds traded at or near par values.

C Priors

Gold denominated yield curve: We use log-normal prior for τ and independent Gaussian priors for the three entries of the initial β vector:

$$\beta_{0,0} \sim \mathcal{N}(10, 10), \qquad \beta_{1,0} \sim \mathcal{N}(10, 10), \qquad \beta_{2,0} \sim \mathcal{N}(10, 10), \qquad \log \tau \sim \mathcal{N}(60, 60)$$

We use weakly informative priors for components of Σ:

- For the standard deviations we use a *common* exponential prior (independent across components) with the rate parameter tuned so that *a priori* the probability that $\sigma_i > 1$ is less than 5%. The mean is 1/3.
- For the correlation matrix Ω we use the LKJ prior with a concentration parameter $\eta = 5$, which is a unimodal but fairly vague distribution over the space of correlation matrices. For η values larger than 1, the LKJ density increasingly concentrates mass around the unit matrix, i.e., favoring less correlation.

Pricing errors: We use *common* exponential priors on the standard deviation of pricing errors, $\sigma_m^{(i)}$, with the rate parameter tuned so that *a priori* the probability that $\sigma_m^{(i)} > 20$ is lower than 5%. The prior mean is 10.

Model of exchange rates: We use independent Gaussian priors for all components of ζ_0 except for F.

- For entries of the initial long-run mean vector μ_0 and matrix K, we set the mean of the Gaussian prior to the point estimates coming from estimating a time-invariant version of model (4.4) using data for $1862 - 1863$. We set the standard deviations so that the prior allows for reasonably large deviations from these point estimates.[40] This procedure guarantees that the prior distribution concentrates on sensible parameter values, but because the estimation is based on a short stretch of data, the location of the parameters is only weakly restricted.
- For entries of the initial persistence matrix A_0 we set a prior that assumes mildly positive auto-correlations for both entries of x_t while being agnostic about the cross-terms.[41] Observe that we do not explicitly restrict A_t to be a stable matrix, but use a prior that pushes the initial A_0 matrix in the direction of the "stable region".
- Parameter matrix F is lower-triangular that we parameterize as follows. First, similar to (5.1), we decompose the covariance matrix FF' into correlation coefficients and marginal variances $FF' = \Xi_F \Omega_F \Xi_F$, where Ξ_F is a diagonal matrix containing the marginal standard deviations and Ω_F is the corresponding correlation matrix. Matrix F can be written as $F = \Xi_F L\Omega_F$, where $L\Omega_F$ is the lower-triangular Cholesky

[39] Chari was responding to the content of a draft version of Hall and Sargent (2011), which documented differences between the US government accounting method and an alternative mark-to-market method.

[40] In particular, we set $\mu_0[1] \sim \mathcal{N}(1, 1)$, $\mu_0[2] \sim \mathcal{N}(1.27, 1)$, and $K[1, 1] \sim \mathcal{N}(0.029, 0.05)$, $K[2, 1] \sim \mathcal{N}(-0.041, 0.05)$, $K[1, 2] \sim \mathcal{N}(0.0, 0.05)$, $K[2, 2] \sim \mathcal{N}(0.025, 0.05)$.

[41] In particular, we set $A_0[1, 1] \sim \mathcal{N}(0.9, 0.1)$, $A_0[2, 1] \sim \mathcal{N}(0, 1)$, $A_0[1, 2] \sim \mathcal{N}(0, 1)$, $A_0[2, 2] \sim \mathcal{N}(0.9, 0.1)$.

factor of Ω_F such that $(L\Omega_F)(L\Omega_F)' = \Omega_F$. For the standard deviations in Ξ_F we use a *common* exponential prior (independent across components) with the rate parameter tuned so that *a priori* the probability that $\sigma_{F,i} > 0.3$ is lower than 5%. The prior mean is 0.1. For the Cholesky factor $L\Omega_F$ we use the LKJ prior with concentration parameter $\eta_F = 5$.

- We assume that Σ_μ and Σ_A are diagonal matrices, i.e., shocks to the components of μ_t and A_t are independent. For their standard deviations we use a *common* exponential prior (independent across components) with the rate parameter tuned so that *a priori* the probability that $\sigma_i > 0.3$ is lower than 5%. The prior mean is 0.1.

D Consistent Estimation of Inflation Expectations

We estimate inflation expectations between 1794-2020 by applying a univariate version of the statistical model we introduced for exchange rate expectations in section 4.3. The underlying data is our combined monthly inflation series that we described in section 2.4. During the temporary suspension of gold convertibility (1862-1879), the General Price Level Index expresses Greenback inflation. We convert this into gold inflation by using the gold/greenback exchange rate P_t.

Let π_{t+1} denote the logarithm of monthly inflation between period t and $t+1$. We model this variable with the following state-space model with (infrequently) changing long-run mean and persistence parameters:

$$\begin{aligned} \pi_{t+1} &= \alpha_t + x_t^\pi + \sigma_\pi \varepsilon_{\pi,t+1} \\ x_{t+1}^\pi &= \rho_t x_t^\pi + \sigma_x \varepsilon_{\pi,t+1} \end{aligned} \qquad \varepsilon_{\pi,t+1} \sim \mathcal{N}(\mathbf{0},1)\,,\ \forall t \geq 0 \tag{D.1}$$

where x_t^π is a hidden state with given initial x_0. Parameters α_t and ρ_t follow random walks with infrequent shocks:

$$\alpha_t = \begin{cases} \alpha_{t-1} + \Sigma_\alpha \varepsilon_{\alpha,t} & \varepsilon_{\alpha,t} \sim \mathcal{N}(\mathbf{0},1) \quad \text{if } \ t = k\Delta \quad \text{for } k \in \mathbb{N} \\ \alpha_{t-1} & \text{otherwise} \end{cases}$$

$$\rho_t = \begin{cases} \rho_{t-1} + \Sigma_\rho \varepsilon_{\rho,t} & \varepsilon_{\rho,t} \sim \mathcal{N}(\mathbf{0},1) \quad \text{if } \ t = k\Delta \quad \text{for } k \in \mathbb{N} \\ \rho_{t-1} & \text{otherwise} \end{cases}.$$

Our baseline estimates set $\Delta = 12$, i.e. the long-run mean and persistence of monthly inflation can change once every year. Model (D.1) posits that j-period ahead logged inflation, $\sum_{i=1}^{j} \pi_{t+i}$, is a normal random variable, implying that j-period ahead gross inflation, $\Pi_t^{(j,n)}$, is lognormal. Using the model-implied conditional mean and variance of $\sum_{i=1}^{j} \pi_{t+i}$, one can derive an estimate for $\mathbb{E}_t\left[\exp\left(-\pi_t^{(j,n)}\right)\right]$ that goes into formula (8.1). We estimate this model using the same HMC-NUTS sampler that we use for our yield curve model.

Priors: We use independent Gaussian priors for σ_x and the initial parameters α_0 and ρ_0:

$$\sigma_x \sim \mathcal{N}(0.01, 0.1), \qquad \alpha_0 \sim \mathcal{N}(1,1), \qquad \rho_0 \sim \mathcal{N}(0.9, 0.1)$$

For the standard deviations σ_π, Σ_α, and Σ_ρ, we use a *common* exponential prior with the rate parameter tuned so that *a priori* the probability that $\sigma_\pi > 1.5$ is lower than 5%. The corresponding prior mean is 0.5.

E Additional Connections Between Spreads and Growth Rates

Table 2 replicates table 1 but uses the change in the spread rather than the level of the spread.

	1797-1860				1866-1933				1950-2000			
	Term spread maturity											
Horizon	10y - 1y		10y - 5y		10y - 1y		10y - 5y		10y - 1y		10y - 5y	
k-years	β_k^1	R^2	β_k^5	R^2	β_k^1	R^2	β_k^5	R^2	β_k^1	R^2	β_k^5	R^2
1-year	0.66	0.071	1.45	0.054	0.09	0.000	0.11	0.000	1.06	0.128	3.81	0.136
	(0.28)		(0.75)		(0.61)		(1.63)		(0.32)		(1.22)	
3-year	1.87	0.109	4.09	0.081	0.24	0.000	1.05	0.001	2.32	0.166	5.99	0.100
	(0.72)		(1.88)		(1.10)		(2.91)		(0.66)		(2.28)	

Table 2: Forecasts of real GDP growth from first differenced term spreads

The table reports the coefficient $\beta_k^{(j)}$ and R^2 for the regression $g_{t+k} = \alpha_k^{(j)} + \beta_k^{(j)} \left(\left(y_t^{(10)} - y_t^{(j)} \right) - \left(y_{t-1}^{(10)} - y_{t-1}^{(j)} \right) \right) + \varepsilon_{t+k,k}^{(j)}$ where g_{t+k} is the annual percentage growth of real GDP over the next k years and $y_t^{(j)}$ denotes the annualized j-year zero coupon yield. We annualize the yields by taking the arithmetic average for each year. Newey and West heteroskedasticity- and autocorrelation-consistent standard errors with lag order one in parentheses. *** 1%, ** 5%, and * 10% significance.

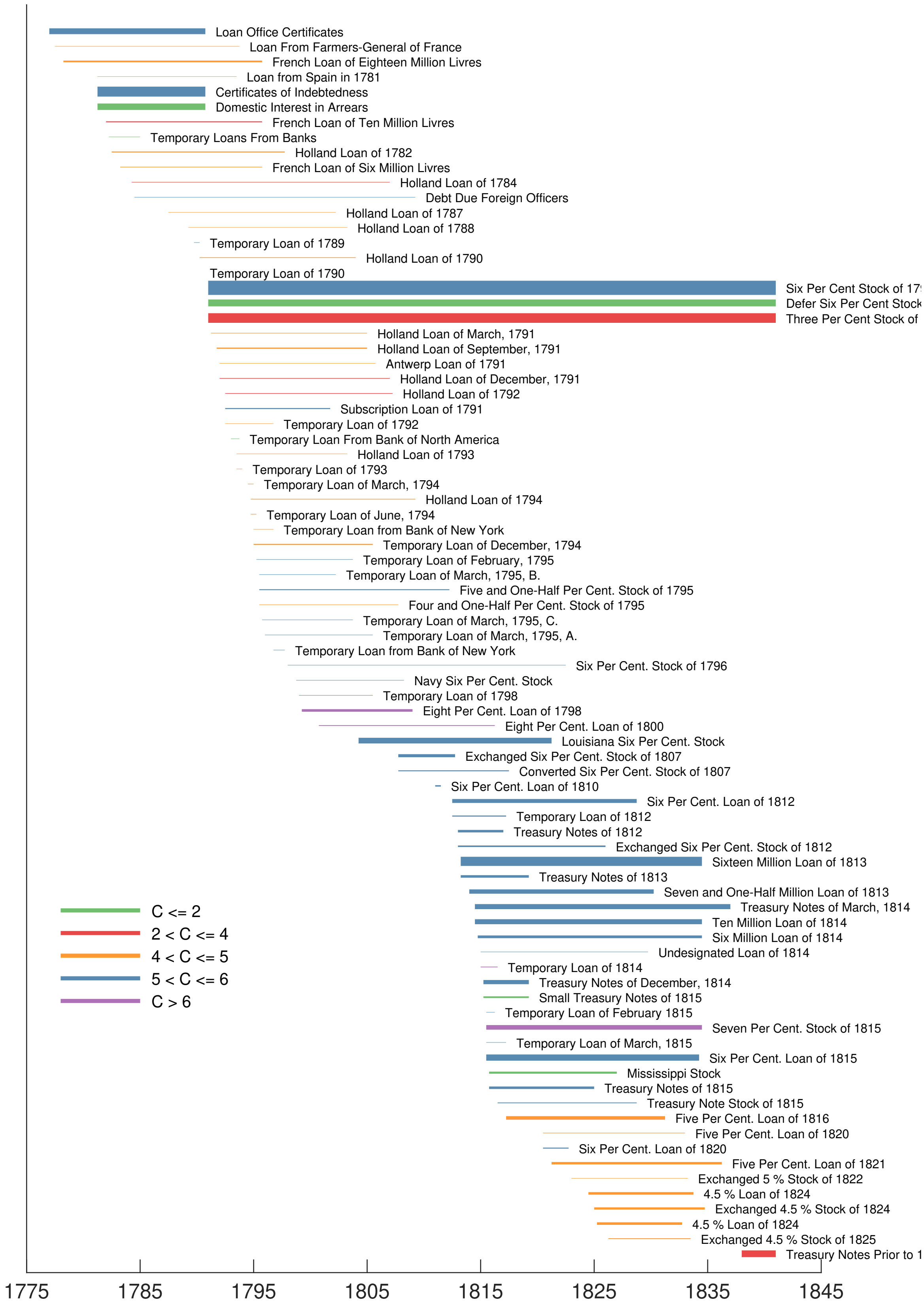

Figure 14: Treasury Bonds Issued from 1776 to 1840.

The span of each line corresponds to the period the security was outstanding. The width is proportional to the size of the issue, and the color denotes the coupon rate.

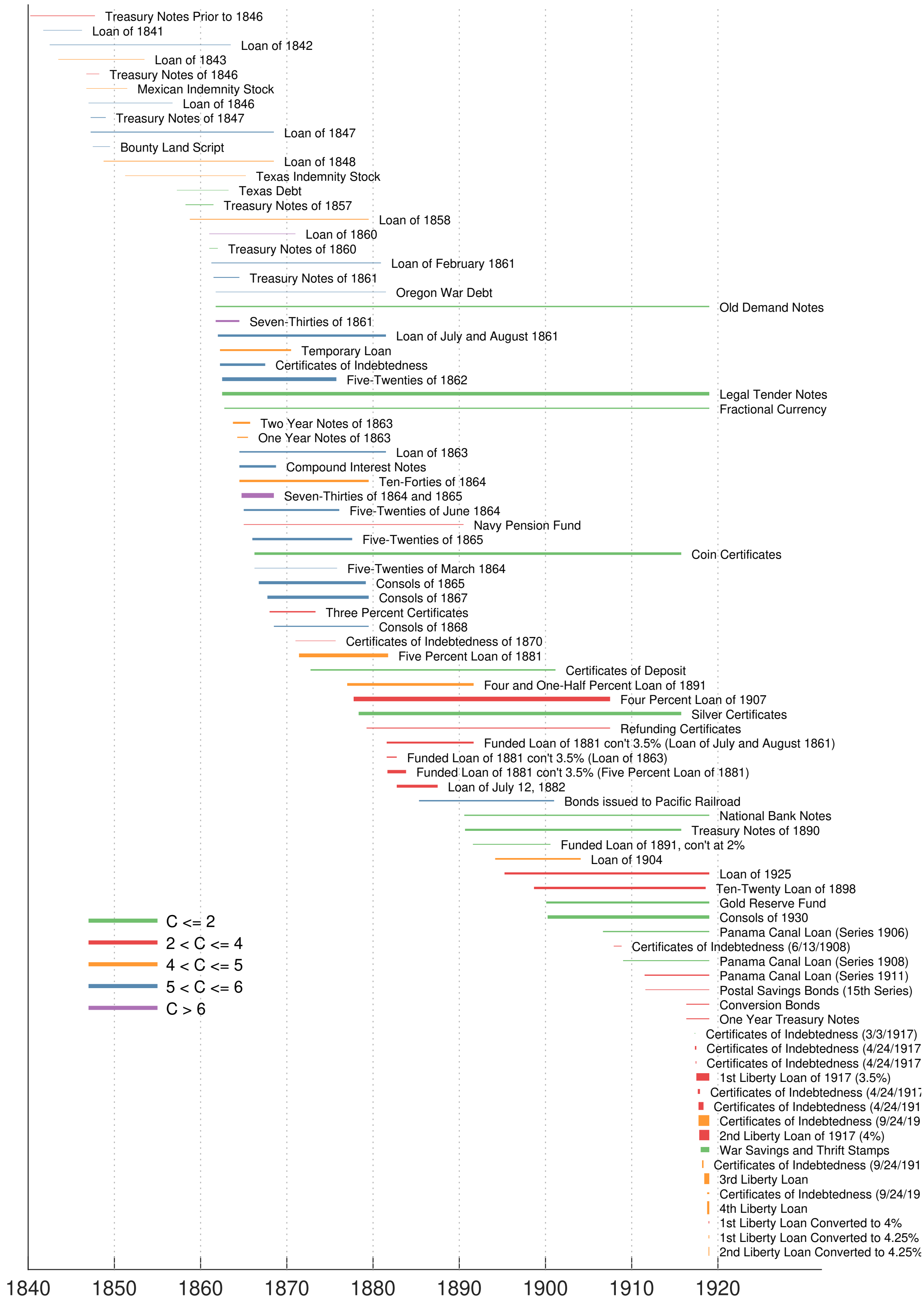

Figure 15: Treasury Bonds Issued from 1840 to 1918.

The span of each line corresponds to the period the security was outstanding. The width is proportional to the size of the issue, and the color denotes the coupon rate.